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ABOUT AN ANALOGY METHOD OF THE NON-LINEAR MATRIX FORM OF UNITARY TRANSFORMATIONS

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ABSTRACT :
In this paper we have explored the various physical and biological model tasks that can be reduced to system the ODE through a non linear matrix. The analysis of the study provides identification internal analogies for which a comparative study has been made with the help of unitary transformation method.
Keywords: ***Internal analogy, Unitary transformations, Stability theory***

1 INTRODUCTION

In the present paper, by the method of internal analogy and algebraic method of unitary transformations some of nonlinear mathematical equations describing physical and biological processes are discussed. The offered methods of the analysis can be very useful when studying a big class of nonlinear tasks of the theory of stability, and also at research of dynamics of the solution of specific nonlinear model objectives that allows to do without device of functions of Lyapunov, keeping efficiency and in critical cases. Matrix form of record of the nonlinear model equations describe some of the physical and biological processes. The method of mathematical modeling is one of the most important methods to study the real world, including the analysis of physical, biological and some other processes. Further the study of model equations in a vector-matrix form provide the method of internal analogy and an algebraic method of unitary transformations for the analysis of stability of the studied processes, including critical cases. The method internal assumes analogies. Data of the studied nonlinear model equations of various physical, social, biological can be expressed in form of matrix. The analysis of the received nonlinear systems the ODE and studying of character of the existing isolated rest points provide identification internal analogies. In the elementary case it is possible to apply Lyapunov's theorem about the analysis of asymptotic stability on the first approach. Existence of internal analogy for the received nonlinear model system allows in certain cases about the method of unitary transformations which are stated in theorems 2, 3, 4 and in works of [5,6].

Existence of internal analogy for the comparative analysis of nonlinear model systems the ODE are found in the following cases:

a) When a nonlinear matrix of $A(z)$ of the studied system the ODE are normal (i.e. in the presence, identical equality of $A(z)A^*(z) \equiv A^*(z)A(z)$ in some area in $\Omega(z)$ [7],

b) Further, when there is a special diagonal matrix of $\bar{Q}(z)$ such that matrix of $B(z) = \bar{Q}(z)A(z)$ is normal,

Here we consider a number of the substantial uncommon model nonlinear differential equations describing various physical and biological processes including critical case:

Example 1. In work [2, page 74] the mechanical system is studied

$$\ddot{x} + a\dot{x} + bx^m = 0; \qquad (a, b > o;\ m = 2k + 1) \tag{1}$$

with one degree of freedom under the influence of the potential nonlinear force and force of resistance, proportional first degree of speed. With use of the theorem of Barbashin - Krasovsky the proof of stability of its decision is provided rather bulky (Lyapunov's theorem is inapplicable here).

We transform the equation (1) to system

$$\begin{cases}\dot{y} = -ay - bx^m; \\ \dot{x} = \qquad y;\end{cases}$$

$$\dot{y} = A(y)y; \quad \left(A(y) = \begin{pmatrix}(-a) & (-bx^{2k}) \\ 1 & 0\end{pmatrix}\right); \quad y = (\dot{x}, x)^T.$$

After non degenerate replacement, we obtain

$$y = S_0 z\,; \quad \left(S_0^{-1}A(0)S_0 = \Lambda_0 = \begin{pmatrix}-a & 0 \\ 0 & 0\end{pmatrix}, \ S_0 = \begin{pmatrix}0 & -a \\ 1 & 1\end{pmatrix}\right), \tag{2}$$

Which yield

$$\dot{z} = B(z)z; \quad [\,B(z) = \begin{pmatrix}-a + q(x) & q(x) \\ -q(x) & -q(x)\end{pmatrix}; \quad q(x) = {bx^{2k}}/{a} > 0]$$

Example 2. Known model equation of fluctuations of a pendulum

$$\ddot{x} + \omega^2 sinx = 0 \tag{3}$$

it can be written down in the form of equivalent system with a nonlinear matrix

$$\dot{y} = A(y)y; \quad A\,(y) = \begin{pmatrix}0 & 1 \\ -a^2(x) & 0\end{pmatrix}; \quad (y = (x, \dot{x})^T, \ \ a^2(x) = \omega^2 \frac{sinx}{x} > 0; \quad |x| < \pi) \tag{4}$$

with two points of rest of $P_0 = (0{,}0)$ и $P_1 = (\pi, 0)$.

In P_0 point we have a critical case as the matrix of $A(P_0)$ has purely imaginary range $\lambda_{Aj} = \pm i\omega$. The point of rest of P_1 will be unstable on the first approach as a range of a matrix of $A(P_0)$ has a range $\lambda_{Aj} = \pm\omega$.

Example 3. The conservative mechanical system with one degree of freedom (without friction) [3, page 175] is described by the model equation:

$$\ddot{x} = f(x); \quad \big(f(0) = 0; \quad f(x) \in C^1(\Omega);\ \Omega{:}\{|x| < R\};\ f(x) = f'(\xi)x\big), \tag{5}$$

which is equivalent to system with a nonlinear matrix

$$\dot{y} = A(y)y; \qquad A\,(y) = \begin{pmatrix}0 & 1 \\ f'(\xi) & 0\end{pmatrix}; \ y = (x, \dot{x})^T. \tag{6}$$

In the presence of a strict minimum of potential energy

$U(x) = -\int_0^x f(t)\,dt$ at point $x = 0$, $(f'(\xi) < 0; \quad |x| < \delta\,)$ the matrix $A\,(0)$ has in $x = 0$ point purely imaginary range $\lambda_{Aj} = \pm\sqrt{f'(\xi)}$ that is here we have a critical case.

Example 4. The simplest model of the genetic mechanism at bacteria is described by model system of the equations. [4, page 55]:

$$\begin{cases}\dot{x} = \frac{p}{y} - a; \\ \dot{y} = qx - b,\end{cases} \quad (a, b, p, q > 0); \tag{7}$$

with a point of rest of $P_0 = (x_0, y_0) = ({b}/{q};\ {p}/{a})$. After replacement $x = x_1 + x_0$; $y = y_1 + y_0$ we will pass to equivalent system with a nonlinear matrix

$$\dot{z} = A(z)z; \ \ (A(z) = \begin{pmatrix}0 & -f(y_1) \\ q & 0\end{pmatrix}; f(y_1) = \frac{a}{y_1 + y_0} > 0, z = (x_1, y_1)^T, \ |y_1| < y_0 \ ), \tag{8}$$

where the matrix of $A(0)$ has in a rest point (0,0) purely imaginary range $\lambda_{Aj} = \pm i\sqrt{qf(y_1)}$, that is here we have a critical case.

Example 5. In biology of the loudspeaker of the interacting populations ("predator victim") it is described by model system of equations Volterra – the Tray [4, page 203]:

$$\begin{cases}\dot{x} = (a - by)x; & (a, b, c, d > 0); \\ \dot{y} = -(c - dx)y;\end{cases} \tag{9}$$

with two points of $P_0(0{,}0)$ & $P_1 = ({c}/{d}\,; {a}/{b})$.

In the first case (at point P_0) we have a saddle point that is an unstable point of rest. In the second case for the analysis of stability of a point of rest of P_1 after replacement

$x = x_1 + x_0$; $y = y_1 + y_0$ we will pass to system

$\dot{z} = A(z)z; \quad \left(A(z) = \begin{pmatrix} 0 & -p(x_1) \\ q(y_1) & 0 \end{pmatrix}; \quad z = (x_1, y_1)^T; \right.$ (10)

$p(x_1) = b\left(x_1 + {}^{c}/_{d}\right) > 0; \; q(y_1) = d\left(y_1 + {}^{a}/_{b}\right) > 0; \quad |x_1| < {}^{c}/_{d}; \quad |y_1| < {}^{a}/_{b}$),

where the matrix of $A(0)$ has purely imaginary range $\lambda_{0j} = \pm i\sqrt{ac}$, that corresponds to a critical case.

We investigate further the system received above with a nonlinear matrix and by means of a method of analogies and algorithm of a method of unitary transformations [5, 6] and we receive criteria of stability of the trivial decision of the systems stated above.

Theorem 1. For a square of Euclidean standard of the decision of system $\dot{x} = A\,(x,t)x$ fairly differential equality $\frac{d|x|^2}{dt} = 2Re(x^*A(x,t)x\,, \;\; (x \in C^n)$.

Proof. Taking into account that

$|x|^2 = x^*x = \sum_1^n x_j^2\,; \quad \dot{x}^* = x^*A^*(x,t); \quad (x = (x_1, \ldots, x_n))^T$ we will write down differential equality:

$$\frac{d|x|^2}{dt} = \frac{d(x^*\,x)}{dt} = \dot{x}^*\,x + x^*\,\dot{x} = x^*A^*(x,t)x + \; x^*A(x,t)x = 2Re(x^*A(x,t)x),$$

Theorem 2 if for Cauchy problem

$$\dot{x} = \; A(x,t)x; \quad x(0) = x^0, \quad (x \in C^n) \qquad (11)$$

With normal matrix $A(x,t)$ [7] $(A^*(x,t)\,A(x,t) \equiv A(x,t)A^*(x,t); \;\; x, t\epsilon D = \{|x| < R; t \geq 0\})$ её спектр $\{\lambda_{Aj}(x,t)\}_1^n$ satisfies in D equation

$$Re\lambda_{Aj}(x,t) \equiv \varphi(t)|x|^\alpha; \qquad (\alpha \geq 0, \quad j = \overline{1,n}), \qquad (12)$$

Then the differential equation $\frac{d|x|^2}{dt} = \varphi(t)|x|^{2+\alpha}$ corresponding to the value

$$|x(t)| = |x^0|exp\,b(t); \qquad (\alpha = 0) \qquad (13)$$

$$|x(t)| = (|x^0|^{-\alpha} - \alpha b(t))^{\frac{-1}{\alpha}}; \quad (\alpha > 0; \quad b(t) = \textstyle\int_0^t \varphi(t)dt \;\;). \qquad (14)$$

Prof. With the help of unitary value

$x = U_A(x,t)y; \quad (|x(t)| \equiv |y(t)|)$

$(U_A{}^*(x,t)A(x,t)U_A(x,t) = \Lambda_A(x,t) = diag\{\lambda_{A1}(x,t), \ldots, \lambda_{An}(x,t)\}$ has differential equation

$\frac{1}{2}\frac{d|x|^2}{dt} = Re(x^*A(x,t)x) = Re(y^*U_A^*(x,t)A(x,t)U_A(x,t)y) = Re(y^*\Lambda_A(x,t)y) =$

$= Re\left(\sum_1^n \lambda_{jA}(x,t)\left|y_j\right|^2\right) \leq \varphi(t)|y|^2|x|^\alpha = \varphi(t)(|x|^2)^{1+\frac{\alpha}{2}}.$

$$\frac{1}{2}\frac{d|x|^2}{dt} = \varphi(t)|x|^2|x|^\alpha,$$

If $\alpha = 0$, тогда $\frac{1}{2}\frac{d|x|^2}{dt} = \varphi(t)|x|^2,$

Let $|x|^2 = z\,,$ т.е. $\frac{1}{2}\frac{dz}{dt} = \varphi(t)z, \int_{z_0}^{z}\frac{dz}{z} = 2b(t)$

$\ln\, |z|_{z_0}^{z} = 2b(t)$

$z = z_0e^{2b(t)}, \;\; |x|^2 = |x_0|^2e^{2b(t)}$ т.е. $x = x_0e^{b(t)}$.

When $\alpha > 0\,;$ $\frac{1}{2}\frac{dz}{dt} = \varphi(t)zz^{\frac{\alpha}{2}}, \quad \int_{z_0}^{z}\frac{dz}{z^{1+\frac{\alpha}{2}}} = 2b(t),$ i.e.

$$\frac{-2}{\alpha}\left(z^{\frac{-\alpha}{2}} - z_0^{\frac{-\alpha}{2}}\right) = 2b(t)$$

$z^{\frac{-\alpha}{2}} - z_0^{\frac{-\alpha}{2}} = -\alpha b(t)$ from this $z(t) = \left(-\alpha b(t) + |z_0|^{\frac{-\alpha}{2}}\right)^{\frac{-2}{\alpha}}.$

Therefore $|x(t)| = (|x_0|^{-\alpha} - \alpha b(t))^{\frac{-1}{\alpha}}$.

if $t \to +\infty \;\; \& \;\; b(t) \to -\infty \qquad x(t) = (x_0^{-\alpha} - \alpha b(t))^{\frac{-1}{\alpha}} \to 0.$

Thus asymptotic stability of the solution of an initial task takes place (11) (at $\alpha = 0 \;\; \& \;\; \alpha > 0$),)

that it was required to prove.

Consequences.

1. The decision of system $\dot{x} = A(x,t)$ with skew-symmetric ($A^T = -A$),or skew-Hermitian ($A^* = -A$),a matrix always steadily, as in this case matrix $A\ (x,t)$, being normal, has purely imaginary range [7], that is

$$Re\lambda_{Aj}(x,t) \equiv 0, \ \left(j = \overline{1,n.}\right).$$

2. Decisions (13) and (14) of system (12) asymptotically are steady at

$$b(t) = \int_0^t \varphi(t)dt \xrightarrow[t\to+\infty]{} -\infty, \text{ and in case } \ b(t) \le C \ \ (\forall t \ge 0) \text{ –of are steady.}$$

Example. The decision of system with a non linear system of matrix

$$\dot{x} = \begin{pmatrix} (a+cost)|x|^{\alpha} & (tx_1x_2) \\ (-tx_1x_2) & (a+cost)|x|^{\alpha} \end{pmatrix} x; \qquad (\alpha \ge 0)$$

asymptotically it is steady at a <0 and it is steady in case of $a = 0$ (owing to a consequence).

Theorem 3. (Analog of the principle of superposition for nonlinear systems).

If system

$$\dot{x} = \big(A(x,t) + B(x,t)\big)x; \quad x(0) = x^0; \quad (x \in R^n) \tag{15}$$

with A, normal in the D matrixes $A(x,t)$ and $B\ (x,t)$ for their range $\{\lambda_{Aj}(x,t)\}_1^n$ & $\{\lambda_{Bj}(x,t)\}_1^n$ are fair in the D ratios.

$$Re\lambda_{Aj}(x,t) = \varphi_A(t); \quad Re\lambda_{Bj}(x,t) = \varphi_B(t); \quad \left(j = \overline{1,n}\right),$$

then for standard of the solution of an initial task (15) the assessment is fair

$$|x(t)| = |x^0| exp\, b(t); \qquad \left(b(t) = \int_0^t (\varphi_A(t) + \varphi_B(t))dt\right).$$

Proof. Taking into account existence of unitary substitutions

$$x = U_A(x,t)y \ , \ x = U_B(x,t)z, \ \ (|x(t)| \equiv |y(t)| \equiv |z(t)|),$$
$$(U_A{}^*(x,t)A(x,t)U_A(x,t) = \Lambda_A(x,t) = diag\{\lambda_{A1}(x,t),\dots,\lambda_{An}(x,t)\};$$
$$U_B{}^*(x,t)B(x,t)U_B(x,t) = \Lambda_B(x,t) = diag\{\lambda_{B1}(x,t),\dots,\lambda_{Bn}(x,t)\}$$

differential equality ($\frac{d|x|^2}{dt} = 2Re(x^*A(x,t)x) + 2Re(x^*B(x,t)x) = 2Re(y^*\Lambda_A(x,t)y) + 2Re(z^*\Lambda_B(x,t)z)$ $= 2(\varphi_A(t) + \varphi_B(t))|x|^2$,takes place as was to be shown.

Consequence. If in the analysis of nonautonomous nonlinear system of a view (15) with two normal matrixes of $A\ (x,t)$ and $B\ (x,t)$, where matrix $B\ (x,t)$ are Cososimmetrichesky or skew-Hermitian (i.e. has purely imaginary range), then the behavior of the decision of system (15) completely is defined by a range of a normal matrix of $A\ (x,t)$.

Theorem 4. If for system

$$\dot{x} = A(x,t)x; \quad x(0) = x^0 \tag{16}$$

exists in the D diagonal matrix $\overline{Q}(x) = diag\{q_1(x_1),\dots,q_n(x_n)\}, \quad (q_j(x_j) > 0; \quad j = \overline{1,n}),$ such that matrix $B(x,t) = \overline{Q}(x)\,A(x,t)$ is in the D normal and its range $\{\lambda_{Bj}(x,t)\}_1^n$ satisfies inequality $Re\lambda_{Bj}(x,t) \le -\beta(x) < 0$, $\left(j = \overline{1,n}\right)$, then the trivial solution of an initial task (16) asymptotically is steady, and in case of случае $Re\lambda_{Bj}(x,t) \le 0 \ \ \left(j = \overline{1,n}, \quad x,t \in \mathrm{D}\right)$ is steady.

Proof follows from existence of function of Lyapunov

$$v(x) = \sum_{j=1}^n \int_0^{x_j} sq_j\,(s)ds > 0; \qquad \left(x_j \ne 0; \ \ j = \overline{1,n}\right) \quad \text{taking into account that}$$

$$\frac{dv}{dt}\Big|_{(16)} = \sum_{j=1}^n \frac{\partial v}{\partial x_j}\,\dot{x}_j = \sum_{j=1}^n x_j q_j\left(x_j\right)\dot{x}_j = x^T\overline{Q}(x)\dot{x} = x^T\overline{Q}(x)A(x,t)x = x^TB(x,t)x =$$

$$\{\text{taking into account } x = U_B(x,t)y\} = y^T\Lambda_B(x,t)y = \sum_{j=1}^n \lambda_{Bj}(x,t)\left|y_j\right|^2 \le -\beta(x)|x|^2,$$

as was to be shown.

By means of the stated method of internal analogy (the Theorem 4) and a method of unitary transformations we investigate the examples reviewed above.

Example 1. For system (2) in an example 1:

$\dot{z} = B(z)z;\ z(0) = z_0 \quad B(z) = \begin{pmatrix} -a + q(x) & q(x) \\ -q(x) & -q(x) \end{pmatrix}, \quad (q(x) = {bx^{2k}}/{a} > 0)$

the matrix of B(z) is expressed in the form of the sum of two normal matrixes

$$B(z) = P(z) + T(z); \qquad (P(z) = \begin{pmatrix} -a + q(x) & 0 \\ 0 & -q(x) \end{pmatrix}, T(z) = \begin{pmatrix} 0 & q(x) \\ -q(x) & 0 \end{pmatrix})$$

From theorem 3 if $(0 < q(x) < a)$ we have asymptotically the steady decision with $(x^{2k} < {a^2}/{b})$, solution.

Example 2. The model equation of fluctuations of a pendulum leads to equivalent system (4) with a nonlinear matrix.

$\dot{y} = A(y)y;\ A(y) = \begin{pmatrix} 0 & 1 \\ -a^2(x) & 0 \end{pmatrix};\ y = (x, \dot{x})^T,\ (a^2(x) = \omega^2 \frac{sinx}{x} > 0; \quad |x| < \pi).$

As there is a diagonal matrix $\overline{Q}(x) = \begin{pmatrix} a^2(x) & 0 \\ 0 & 1 \end{pmatrix}$ such that a matrix

$B(x) = \overline{Q}(x)A(y) = \begin{pmatrix} 0 & a^2(x) \\ -a^2(x) & 0 \end{pmatrix}$ owing to the theorem 4 zero point of rest of system (4) and the initial equation (3) will be steady as the normal matrix of $B(x)$ has purely imaginary range.

Example 3. Research of conservative mechanical system (5) leads to equivalent system (6) $\dot{y} = A(y)y;\ A(y) = \begin{pmatrix} 0 & 1 \\ f'(\xi) & 0 \end{pmatrix}; \quad (f'(\xi) < 0)$ Owing to existence of a diagonal matrix $\overline{Q}(x) = \begin{pmatrix} -f'(\xi) & 0 \\ 0 & 1 \end{pmatrix}$ and $B(y)$ normal matrix alit $B(y) = \overline{Q}(x)A(y) = \begin{pmatrix} 0 & -f'(\xi) \\ f'(\xi) & 0 \end{pmatrix}$ which, being normal, has purely imaginary range $\lambda_{Bj} = \pm i f'(\xi)$, therefore the point of rest of $x = 0$ will be steady.

Example 4. As the analysis of the genetic mechanism at bacteria can be described by model system (8):

$\dot{z} = A(z)z; \quad A(z) = \begin{pmatrix} 0 & -f(y_1) \\ q & 0 \end{pmatrix}; (f(y_1) = \frac{a}{y_1 + y_0} > 0)$, then existence of a diagonal matrix

$\overline{Q}(z) = \begin{pmatrix} q & 0 \\ 0 & f(y_1) \end{pmatrix}$ and normal matrix

$B(z) = \overline{Q}(z)A(z) = \begin{pmatrix} 0 & -qf(y_1) \\ qf(y_1) & 0 \end{pmatrix}$ with purely imaginary range $\lambda_{Bj} = \pm iqf(y_1)$ guarantees stability of a point of rest of $P_0 = \left({b}/{q},\ {p}/{a}\right)$

Example 5. The analysis of biological model (9) of Voltaire – the Tray we reduced to research of equivalent system (10) with a nonlinear matrix:

$$\dot{z} = A(z)z;\ A(z) = \begin{pmatrix} 0 & -p(x_1) \\ q(y_1) & 0 \end{pmatrix}; (z = (x_1, y_1)^T;\ p(x_1) = b\left(x_1 + \frac{c}{d}\right) > 0;$$

$q(y_1) = d\left(y_1 + \frac{a}{b}\right) > 0$). By analogy we will find a diagonal matrix

$\overline{Q}(z) = \begin{pmatrix} p^{-1}(x_1) & 0 \\ 0 & q^{-1}(y_1) \end{pmatrix}$ the normal of matrix $B(z)$ allowing to construct $B(z) = \overline{Q}(z)A(z) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ with purely imaginary range $\lambda_{Bj} = \pm i$, that guarantees stability of a point of rest of $P_1 = \left({c}/{d},\ {a}/{b}\right)$

Example 6. In the analysis nonautonomous nonlinear systems [2, page 68]

$$\dot{x} = (Esint + B(x,t))x; \quad (x \in R^n), \tag{17}$$

(where $B(x,t)$ a cososimmetrichesky matrix), application of a method of functions of Lyapunov in a standard form $v(t) = |x|^2$ doesn't yield result as its derivative owing to system $\dot{v}|_{(17)} = 2vsint$ znakopostoyanny Stability trivial the decision in work [2] is proved by means of rather bulky method of comparison.

The same result can be received more simply by means of the theorem 3 (and consequences to it) as matrixes of Esint and $B(x,t)$ are normal. Taking into account that matrix $B(x,t)$ has purely imaginary range, the decision of system (17) will be steady because the range of a matrix of Esint meets a condition $\int_0^t sin\tau d\tau \leq C$.

Example 7. Nonlinear system [2, page 20]

$$\begin{cases} \dot{x} = -ay + bx\rho; \\ \dot{y} = ax + cy\rho; \end{cases} \quad \left(\rho = \sqrt{x^2 + y^2}\right)$$

in the vicinity of a zero point of rest of $P_0 = (0,0)$it can be written down in a quasilinear form

$$\dot{z} = (A(z) + B)z; \tag{18}$$

with a normal of matrix $A(z) = \begin{pmatrix} b\rho & 0 \\ 0 & c\rho \end{pmatrix}$ and a kososimmetrichesky matrix of $B = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}$ (having purely imaginary range $\lambda_{jB} = \pm i|a|$).Therefore owing to the theorem 3 (and consequences) the trivial decision of system (18) the asymptotic is steady against it at *b,c<0,* is steady at *b = c = 0,* and in case of *b, c > 0* is unstable.

CONCLUSION

We find that various physical and biological model tasks can be reduced to systems the ODE with a nonlinear matrix. The subsequent application of a method of internal analogy and a method of unitary transformations allows to prove on uniform constructive algorithm stability of the studied rest points without the use of the device of functions of Lyapunov. The offered method of the analysis allows to solve problems of the specified type in the course of scientific research work.

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ON WEIGHTED VECTORIAL DYNAMIC INTUITIONISTIC FUZZY OPERATOR OF TYPE s, s $\neq$ 0

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ABSTRACT :
In the present communication, we have defined weighted vectorial dynamic intuitionisitc fuzzy operator of type s, s $\neq$ 0. This has been characterized through basic unit-interval monotonic function (BUMF) and difference equation. Using BUMF, we can obtain the weight vector.

1. INTRODUCION

Zadeh [14] introduced fuzzy set theory in 1965. Thereafter, a lot of work has been initiated by many researchers. Atanassov [1] generalized fuzzy in terms of intuitionistic fuzzy set (IFS) and characterised it by membership function and non-membership function and hesitation degree function. IFS has proven to be highly useful to deal with uncertainty and vagueness. A lot of work has been done to develop and enrich IFS theory [4]. In many complex decision making problems the decision information provided by the decision makes it often imprecise or uncertain due to time pressure, lack of data or the decision maker's limited attention and information processing capabilities. Therefore, IFS is a very highly suitable tool to be used to describe the imprecise or uncertain decision information.

Recently, Gau and Buehrer [8] introduced the vague set, which is equivalent to IFS [6]. Later, based on vague set, Chen and Tan [6] and Hong and Choi [9] utilized the minimum and maximum operations to develop some approximated technique for handling multi-attribute decision making problems under fuzzy environment. Atanassov et al. [3] proposed an intuitionistic fuzzy interpretation of multi-person multi-attribute decision making, in which each decision maker is asked to evaluate at least a part of the alternatives in terms of their performance with respect to each predefined attribute : the decision maker's evaluations are expressed in a pair of numeric values, interpreted in the intuitionistic fuzzy framework: these numbers express a 'positive' or a 'negative' evaluation, respectively. They also proposed a method for multi-person multi-attribute decision making and prescribed some examples of the proposed method in context of public relation and mass communication.

Xu and Yager [13] developed some aggregation operators indicating the intuitionistic fuzzy weighted geometric operators, which extend the traditional weighted geometric operator and then intuitionistic fuzzy hybrid geometric operator which extend the traditional weighted geometric operator and ordered weighted geometric operator to accommodate the environment where the arguments are IFS. Liu and Wang [10] gave an evaluation function for the decision making problem to measure the degree to which alternatives satisfy and do not satisfy the decision maker's requirement. Then they introduced the intuitionistic fuzzy point operators, and defined a series of new score function for the multi-attribute decision making problems based on intuitionistic fuzzy point operators and evaluation functions. Xu [11] defined some new intuitionistic preference relations, such as consistent intuitionistic preference relation, incomplete intuitionistic preference relation and acceptable intuitionistic preference relation and studied their properties.

All these studies are focussed on the decision making problems where all the decision making information are provided at the same period. However, in many cases of decision such as multi-period investment, medical diagnosis, personal dynamic examination and the military system efficiency dynamic evaluation, etc. The original decision information are usually collected at different periods. Thus it is necessary to develop some approaches to deal with these issues.

In the present communication, we shall study the intuitionistic fuzzy multi-attribute decision making problem, where all the attribute values are expressed in intuitionistic fuzzy numbers collected at different periods and these are called dynamic intuitionistic fuzzy multi-attribute decision making (DIF-MADM) problems.

To tackle such problems, we first introduce the notion of intuitionistic fuzzy variable and develop an aggregation operator called 'weighted vectorial dynamic intuitionistic fuzzy averaging (WVDIFA) operator. Then we generalize it in terms of 'Weighted Victorial dynamic intuitionistic Fuzzy averaging operator of type s,' s ≠ 0. Then we use the BUM – Basic unit –interval monotonic method, the difference equations to determine the weight vectors associated with the operator.

SECITON-2

2.1 BASIC CONCEPTS

Definition 1: [14] Let a set Z be fixed, a fuzzy set F in Z is given by Zadeh [14] as follows:

$$F=\left\{\left\langle z,\ \mu_F(z)\right\rangle /\ z\in Z\right\} \tag{1}$$

where

$$\mu_F : Z\rightarrow[0,1],\ z\in Z\rightarrow, \mu_z(z)\in[0,1] \tag{2}$$

and $\mu_F(z)$ denotes the degree of membership of the element z ∈ Z.

Definition 2: [1], Let a set Z be fixed and IFS A ⊂ Z is given by Atanassov [1] is an object having the following form:

$$A=\left\{\left\langle z,\ \mu_A(z),\ \nu_A(z)\right\rangle \setminus\ z\in Z\right\} \tag{3}$$

where the functions:

$$\mu_A : Z\rightarrow[0,1],\ \ z\in Z\rightarrow \mu_A(z)\in[0,1] \tag{4}$$

and

$$\nu_A : Z\rightarrow[0,1],\ \ z\in Z\rightarrow \nu_A(z)\in[0,1] \tag{5}$$

with the condition

$$0\le \mu_A(z)+\nu_A(z)\le 1 \quad \forall\ z\in Z \tag{6}$$

$\mu_A(z),\ \nu_A(z)$ denote the degree of membership and the degree of non-membership of the element z ∈ Z to the set A, respectively. In addition, for each IFS $A\subset Z$, if

$$\pi_A(z)=1-\mu_A(z)-\nu_A(z)\,. \tag{7}$$

Then $\pi_A(z)$ is called the degree of indeterminant of z to A [z] or called degree of hesitation of z to A [1]. Especially, if $\pi_A(z)=0, \quad \forall\ z\in Z$ then IFS A is reduced to a fuzzy set.

Let $\alpha_1=(\mu_\alpha,\ \nu_\alpha,\ \pi_\alpha)$ be an intuitionistic number (IFN), where $\mu_\alpha\in[0,1]$, $\nu_\alpha\in[0,1]$ and

$$\pi_\alpha=1-\mu_\alpha-\nu_\alpha,\ 0\le\mu_\alpha+\nu_\alpha\le 1 \tag{8}$$

For an IFN $\alpha=(\mu_\alpha,\nu_\alpha,\pi_\alpha)$, if the value μ_α is bigger and the value ν_α is smaller than the IFN α gets greater and hence from (8), we have $\alpha^+=(1,\ 0,\ 0)$ and $\alpha^-=(0,\ 1,\ 0)$ are the largest and the smallest IFN respectively.

Definition 3. Hamming Distance Between IFSs

Let $\alpha_1=(\mu_{\alpha_1},\nu_{\alpha_1},\pi_{\alpha_1})$, $\alpha_2=(\mu_{\alpha_2},\nu_{\alpha_2},\pi_{\alpha_2})$ be the IFNs then

$$d(\alpha_1,\alpha_2)=\frac{1}{2}\left[\,|\,\mu_{\alpha_1}-\mu_{\alpha_2}\,|+|\,\nu_{\alpha_1}-\nu_{\alpha_2}\,|+|\,\pi_{\alpha_1}-\pi_{\alpha_2}\,|\,\right] \tag{9}$$

Definition 4 : Hamming Distance of type s, s ≠ 0

$$d^s(\alpha_1,\alpha_2)=\frac{1}{2}\left[|\mu_{\alpha_1}-\mu_{\alpha_2}|^s+|\nu_{\alpha_1}-\nu_{\alpha_2}|^s+|\pi_{\alpha_1}-\pi_{\alpha_2}|^s\right] \tag{10}$$

SECITON 3

3.1 THE WEIGHTED VECTRIAL INTUITIONISTIC FUZZY OPERATOR OF TYPE s, s ≠ 0

Atanssaov [1] defined some basic operations and relations over IFSs. De et al. [7] adding some new operations such as concentration and normalization of IFS, characterized IFS's. Xu and Yager [13] developed some geometric operators to aggregate intuitionistic fuzzy information. All these operations, relations and operators have been used to deal with time independent arguments. If time is taken into account, for examples, the argument information may be collected at different time periods, then the aggregation operators and their associated weights should not be kept constant. So first we define the notion of intuitionistic fuzzy variable.

Definition 5: Let t be a time variable, then we call $\alpha(t)=(\mu_{\alpha(t)},\,\nu_{\alpha(t)},\,\pi_{\alpha(t)})$ an intuitionistic fuzzy variable, where

$$\mu_{\alpha(t)}\in[0,1],\quad \nu_{\alpha(t)}\in[0,1],$$
$$0\le\mu_{\alpha(t)}+\nu_{\alpha(t)}\le 1$$
$$\pi_{\alpha(t)}=1-\mu_{\alpha(t)}+\nu_{\alpha(t)} \tag{11}$$

For an intuitionistic fuzzy variable

$\alpha(t)=(\mu_{\alpha(t)},\ \nu_{\alpha(t)},\,\pi_{\alpha(t)})$, if

$t=t_1,\,t_2,\,t_3,....t_s$ then $\alpha(t_1),\,\alpha(t_2),\,\alpha(t_3),....\alpha(t_s)$ indicates p IFN's collected at p different periods.

Definition 6: Weighted Vectorial Dynamic Intuitionistic Fuzzy Averaging Operator of Type s, s ≠ 0

Let $\alpha_s(t_1)=\mu_{\alpha_s}(t_1),\,\nu_{\alpha_s}(t_1),\pi_{\alpha_s}(t_1))$

and $\alpha_s(t_2)=\mu_{\alpha_s}(t_2),\,\nu_{\alpha_s}(t_2),\pi_{\alpha_s}(t_2))$

be two IFNs, then

(i) $\alpha_s(t_1)\otimes\alpha_s(t_2)$

$$(\mu_{\alpha_s(t_1)}+\mu_{\alpha_s(t_2)}-\mu_{\alpha_s(t_1)}\mu_{\alpha_s(t_2)},\,\nu_{\alpha_s(t_1)}\nu_{\alpha_s(t_2)},\\ (1-\mu_{\alpha_s(t_1)})(1-\mu_{\alpha_s(t_2)})-\nu_{\alpha_s(t_1)}\nu_{\alpha_s(t_2)}) \tag{12}$$

(ii) $$\lambda_{\alpha_s}(t_1)=(1-(1-\mu_{\alpha_s(t_1)})^\lambda,\,\nu_{\alpha_s}^\lambda(t_1),\,(1-\mu_{\alpha_s}(t_1))^\lambda-\mu_{\alpha_s}^\lambda(t_1)),\,\lambda>0 \tag{13}$$

is a weighted vectorial dynamic intuitionistic fuzzy averaging operator of type s, s ≠ (WVDIFA).

Definition 7: Let $\alpha_s(t_1),\,\alpha_s(t_2),\,\alpha_s(t_3),....,\,\alpha_s(t_p)$ be a collection of IFNs collected at p different periods, $t_k(k=1,2,..,p)$ and $\lambda_s(t)=(\lambda_s(t_1),\,\lambda_s(t_2),....\lambda_s(t_p))^T$ be the vectorial weights for periods $t_k(k=1,2,3,...,p)$, then we call WVDIFA $\lambda_s(t)(\alpha_s(t_1),\,\alpha_s(t_2),.....\alpha_s(t_p))$ =

$$\lambda_s(t_1)\alpha_s(t_1)\otimes\lambda_s(t_2)\otimes\lambda_s(t_2)\alpha_s(t_3)\otimes....\otimes\lambda_s(t_p)\alpha_s(t_p) \tag{14}$$

a vectorial weighted dynamic intuitionistic fuzzy averaging operator of type **s, s ≠ 0.**

Recalling (13), we have

VWDIFA $\lambda_{s(t)}(\alpha_s(t_1),\,\alpha_s(t_2),....\alpha_s(t_p))$

$$= \left(\left(1 - \prod_{k=1}^{p} (1 - \mu_{\alpha_s(t_k)} \right)^{\lambda_s(t_k)}, \left(\prod_{k=1}^{p} (\nu_{\alpha_s(t_k)} \right)^{\lambda_s(t_k)}, \right.$$

$$\left. \left(\prod_{k=1}^{p} (1 - \mu_{\alpha_s(t_k)} \right)^{\lambda_s(t_k)} - \left(\prod_{k=1}^{p} \nu_{\alpha_s(t_k)} \right)^{\lambda_s(t_k)} \right) \quad (15)$$

where

$$\lambda_s(t_k) \geq 0, \ \forall k = 1, 2, ..., .p, \ s \neq 0, \text{ and } \sum_{k=1}^{p} \lambda_s(t_k) = 1 \quad (16)$$

SECTION 4

4.1 DETERMINATION OF VECTORIAL WEIGHT $\lambda_s(t)$ OF THE PERIODS t_k (k = 1, 2, ..., p) of type s, $s \neq 0$

Method 1: BASIC UNIT-INTERVAL MONOTONIC METHOD [18]

Let

$$\lambda_s(t_k) = Q_s\left(\frac{k}{p}\right) - Q_s\left(\frac{k-1}{p}\right) \quad (17)$$

For example if $Q_s(x) = x^{r-s+1}, \ r > 0, \ r > s, \ s \neq 0,$

$$\lambda_s(t_k) = \left(\frac{k}{p}\right)^{r-s+1} - \left(\frac{k-1}{p}\right)^{r-s+1} \quad (18)$$

$$= \left(\frac{k}{p}\right)^{r-s+1} - \left(\frac{k}{p} - \frac{1}{p}\right)^{r-s+1},$$

Let $\dfrac{k}{p} = x$.

$$f(x) = x^{r-s+1} - \left(x - \frac{1}{p}\right)^{r-s+1}, x \geq \frac{1}{p}, \ s \neq 0 \quad (19)$$

$$\Rightarrow \qquad f'(x) = (r-s+1)x^{r-s} - (r-s+1)\left(x - \frac{1}{p}\right)^{r-s}$$

$$= (r-s+1)\left[x^{r-s} - \left(x - \frac{1}{p}\right)^{r-s}\right]$$

Thus if r > 1, r – s > 0 (20)

Hence from (17) , we have

(i) r > 1 $\lambda_s(t_{k+1}) > \lambda_s(t_k)$, k = 1, 2, ..., p. The sequence $\{\lambda_s(t_k)\}$ is monotonic increasing sequence, in particular, where r = s and s = 1, then

$$\lambda_s(t_{k+1}) - \lambda_s(t_k)$$

$$= \left(\frac{k+1}{p}\right)^2 - \left(\frac{k}{p}\right)^2 - \left(\frac{k}{p}\right)^2 + \left(\frac{k-1}{p}\right)^2 \quad (21)$$

$$= \frac{2}{p}, \quad k = 1, 2, ..., p-1. \tag{22}$$

Here (22) is a difference equation of first order onto first degree. Also the sequence $\{\lambda_s(t_k)\}$ is an increasing arithmetic sequence:

(ii) When r = 1, s = 0

$$\lambda_0(t_k) = \left(\frac{k}{p}\right)^2 - \left(\frac{k-1}{p}\right)^2$$

$$= \left(\frac{2k-1}{p^2}\right) \tag{23}$$

$$\lambda_s(t) = \frac{1}{p^2}$$

$$\lambda_s(t) = \left(\frac{1}{p^2}, \frac{1}{p^2}, ..., \frac{1}{p^2}\right)^T \text{, which is a constant sequence.} \tag{24}$$

(iii) When r < 1, r – s < 1

then

$$\lambda_s(t_{k+1}) < \lambda_s(t_k), \qquad k = 1, 2, ..., b$$

The sequence $\{\lambda_s(t_k)\}$ is monotonic decreasing sequence.

Method 2: Application of Difference equation

Consider the difference equation (22)

$$\lambda_s(t_{k+1}) - \lambda_s(t_k) = \frac{1}{k^2} \tag{25}$$

This is difference equation of first order and first degree so

C.S. = m – 1 = 0 ⇒ m = 1

Hence C.S. = c_1 (25)

Now $$P.S. = \lambda_s(t_k) = \frac{1}{(E-1)}\frac{1}{K^2}$$

$$= E^{-1}(1 - E_2^{-1})\frac{1}{k^2}$$

$$= (E^{-1} + E^{-2} + E^{-3} + ...)\frac{1}{k^2}$$

$$= \frac{1}{(k-1)^2} + \frac{1}{(k-2)^2} + \frac{1}{(k-3)^2} +$$

Hence the G.S. = $c_1 + \frac{1}{(k-1)^2} + \frac{1}{(k-2)^2} + \frac{1}{(k-3)^2} + ...$

Now $$\lambda_s(t_k) = c_1 + \frac{1}{(k-1)^2} + \frac{1}{(k-2)^2} + \frac{1}{(k-3)^2} + ... \tag{26}$$

which is an increasing monotonic sequence

Hence $\lambda_s(t_k)$, the weighted vector has been obtained.

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COMPARISON OF PROFIT OF A TWO-UNIT COLD STANDBY SYSTEM WITH INSTRUCTION TIME

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ABSTRACT :
The comparison of profit of a two-unit cold standby system with instruction time have been analysed. We have taken a two-unit cold standby system having an expert repairman and his assistant. The assistant repairman can repair a failed unit only after getting instructions from the expert. If the expert repairman is engaged in repairing a failed unit and at that time second unit fails then he leaves the repair in the state of temporary suspension and starts giving instructions to his assistant for repairing the second unit. Also the profit is calculated for a particular case where the repair time and instruction time distributions are exponential. Graphs are plotted and the model is compared with another model in which both the repairman work independently.

INTRODUCTION

In the light of various assumptions regarding failures, repairs, inspections, etc. many investigations concerning reliability of two-unit redundant systems operating under different environments have been made. Goyal and Murari [1] analysed a two-unit standby system with two types of repairman. Sinha and Kapil [2] studied a two-unit redundant system with switching device and two types of repair. Naidu and Gopalan [3] investigated the stochastic behaviour of a two-unit system with one server and introduced the concept of non-negligible inspection time. Tuteja et al. [4] carried out the cost-benefit analysis of two-unit system with partial failures and three types of repair. There may also be situations in which if a unit fails, we call an expert repairman to repair the failed unit who comes with his assistant. The assistant repairman repairs the failed unit perfectly only on the instructions given by the expert.

Thus, introducing the concept of instruction time we, in this paper, investigate a two-unit cold standby system with an expert repairman and his assistant. The instruction time is that for which the expert gives the instructions to his assistant. If a unit fails, the expert repairman is called immediately to repair failed unit. He come to the system with his assistant repairman. If the expert repairman is busy in repairing a unit and second unit fails then he leaves the repair of the first unit and starts to give instructions to his assistant for repairing the second unit. Instruction time is taken as a random variable. On the completion of the instruction time expert repairman resumes the repair of the unit which was under his repair. The assistant repairman takes the second unit (for which he has got instructions) under his repair. It is assumed that the assistant repairman repairs the failed unit perfectly after getting instructions by the expert.

The system is analysed by making use of semi-Markov processes and regenerative processes ; the various measures of system effectiveness such as mean time to system failure (MTSF), steady-state availability, total fraction of busy time for the expert repairman, expected no. of calls of the expert repairman and expected profit incurred to the system are determined. Also the profit is calculated for a particular case where the repair time and instruction time distributions are exponential. Graphs are plotted and the model is compared with another model in which both the repairman work independently.

DESCRIPTIONS OF MODELS AND ASSUMPTIONS

(i) The system consists of two identical units. Initially one unit is operative and the other is cold-standby.
(ii) Initially one unit starts operating, while the other is kept as standby. The standby unit can not fails.
(iii) Failures are self-announcing.
(iv) The system becomes inoperable on the failure of both the units.
(v) Switching for going to operating state from standby state is perfect and instantaneous
(vi) Priority for repair is given to the expert.
(vii) Expert has to leave the repair in state of temporary suspension when instructions are required.
(viii) The assistant repairman repairs the failed unit perfectly on the instructions given to him by the expert.
(ix) The instruction time is arbitrary distributed.
(x) The payment is made only to the expert repairman. He pays himself to his assistant.
(xi) Each unit has an exponential distribution of time to failure while distribution of repair time for the expert repairman is arbitrary and that of his assistant is exponential.
(xii) Failures of a unit are detected immediately and perfectly.
(xiii) After any repair, a unit works like a new one.
(xiv) All the random variables are independent.

NOTATIONS

λ constant failure rate of an operative unit.
$g(t)$, $G(t)$ p.d.f. and c.d.f. of time to repair by expert.
μ constant repair rate of assistant repairman.
$i(t)$, $I(t)$ p.d.f. and c.d.f. of time when expert gives instructions to his assistant.

Other symbols which are used in this paper may seen in Ref. [4]. A transition diagram showing the various rate of transition of the system is given by Fig. 1.

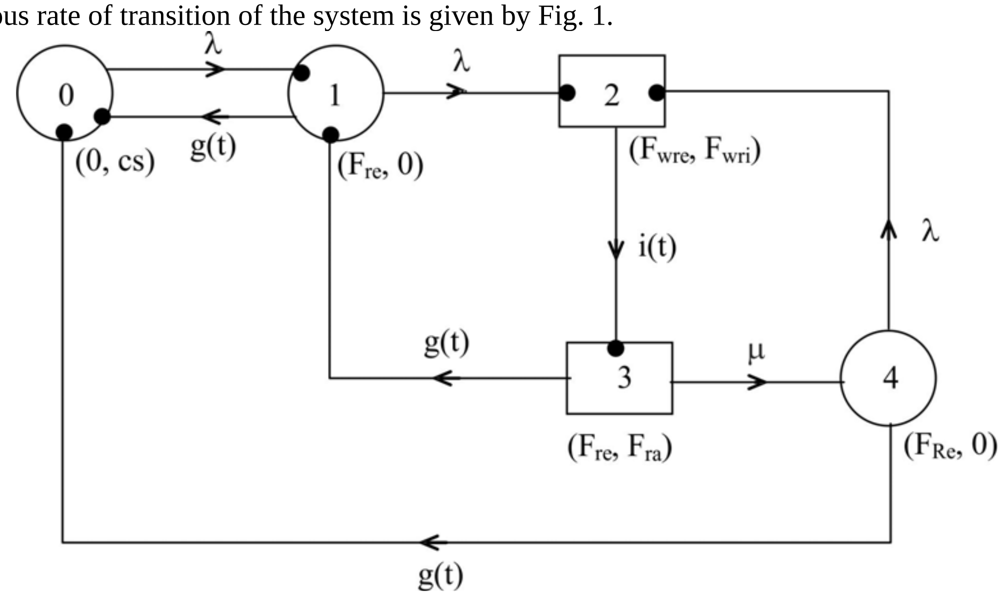
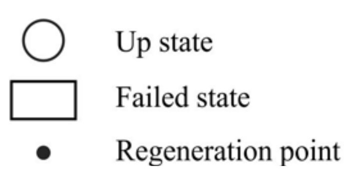


Fig. 1 State Transition Diagram for Model - 1



The symbol in the transition diagram are :-

o operative

cs cold standby

F_{re} failed and under repair of expert repairman

F_{wri} failed and waiting for repair while expert gives instructions to his assistant

F_{wre} failed and waiting for repair of expert

F_{ra} failed and under repair of assistant repairman

F_{Re} failed and repair is continued from earlier state by the expert repairman.

The epochs of entry into states 0, 1, 2 and 3 are regeneration points and thus 0, 1, 2 and 3 are regenerative states. States 2 and 3 are failed states.

MEAN TIME TO SYSTEM FAILURE ANALYSIS

$$\phi_0(t) = Q_{01}(t)\,(s)\,\phi_1(t)$$

$$\phi_1(t) = Q_{10}(t)\,(s)\,\phi_0(t) + Q_{12}(t) \qquad \ldots(1\text{-}2)$$

where the transition probabilities are :

$q_{01}(t) = \lambda e^{-\lambda t}$; $q_{10}(t) = e^{-\lambda t} g(t)$;

$q_{12}(t) = \lambda e^{-\lambda t}$; $q_{23}(t) = i(t)$;

$q_{31}(t) = e^{-\mu t} g(t)$;

$$q_{32}^{(4)}(t) = [\mu e^{-\mu t} © \lambda e^{-\lambda t}]\overline{G}(t) = \frac{\mu\lambda[e^{-\mu t} - e^{-\lambda t}]\overline{G}(t)}{(\lambda-\mu)} ;$$

$$q_{30}^{(4)}(t) = [\mu e^{-\mu t} © e^{-\lambda t}]\, g(t) = \frac{\mu[e^{-\mu t} - e^{-\lambda t}]g(t)}{(\lambda-\mu)} . \qquad \ldots(3\text{-}9)$$

$$\lim_{s\to 0} q_{ij}{*}(s) = p_{ij} . \qquad \ldots(10)$$

$p_{01} = p_{23} = 1$; $p_{10} = g^*(\lambda)$; $p_{12} = 1 - g^*(\lambda)$

$p_{31} = g^*(\mu)$; $p_{32}^{(4)} = \dfrac{1-(\lambda g^*(\mu) - \mu g^*(\lambda))}{(\lambda-\mu)}$

$$p_{30}^{(4)} = \frac{\mu[g^*(\mu) - g^*(\lambda)]}{(\lambda-\mu)} \qquad \ldots(11\text{-}16)$$

By these transition probabilities, it can be verified that

$$p_{01} = p_{10} + p_{12} = p_{23} = p_{31} + p_{32}^{(4)} + p_{30}^{(4)} = 1 \qquad \ldots(17)$$

The mean sojourn times are :

$\mu_0 = \int P(T > t)\,dt = \dfrac{1}{\lambda}$, $\mu_1 = \dfrac{1-g^*(\lambda)}{\lambda}$,

$\mu_2 = \int I(t)\,dt = \int t\,dI(t)$, $\mu_3 = \dfrac{1-g^*(\mu)}{\mu}$, …(18-21)

The unconditional mean time taken by the system to transit for any regenerative state i, when it (time) is counted from the epoch of entrance into that state is mathematically stated as :

$$m_{ij} = \int t\,dQ_{ij}(t) = -\frac{d}{ds} Q_{ij}{*}(s)\Big|_{s=0} \qquad \ldots(22)$$

Thus,

$$m_{01} = \mu_0,\ m_{10} + m_{12} = \mu_1,\ m_{23} = \mu_2\ .$$

$$m_{31} + m_{32}^{(4)} + m_{30}^{(4)} = \frac{\lambda^2(1 - g^*(\mu)) - \mu^2(1 - g^*(\lambda))}{\lambda\mu(\lambda - \mu)} = k \text{ (say)} \qquad \text{…(23-26)}$$

Taking Laplace steltjes transform of relations given in (1-2) and solving for $\phi_0^{**}(s)$, and then in steady state the mean time to system failure is given by :

$$T_0 = \lim_{s\to 0} \frac{1 - \phi_0^{**}(s)}{s} = \frac{\mu_0 + \mu_1}{p_{12}} \qquad \text{…(27)}$$

AVAILABILITY ANALYSIS

$$AV_0(t) = M_0(t) + q_{01}(t) © AV_1(t)$$
$$AV_1(t) = M_1(t) + q_{10}(t) © AV_0(t) + q_{12}(t) © AV_2(t)$$
$$AV_2(t) = q_{23}(t) © AV_3(t)$$
$$AV_3(t) = M_3(t) + q_{31}(t) © AV_1(t) + q_{30}^{(4)}(t) © AV_0(t) + q_{32}^{(4)}(t) © AV_2(t) \qquad \text{…(28-31)}$$

where

$$M_0(t) = e^{-\lambda t}$$
$$M_1(t) = e^{-\lambda t}\ \overline{G}(t) \text{ and } M_3(t) = [\mu\ e^{-\mu t} © e^{-\lambda t}]\ \overline{G}(t)$$

Taking Laplace transform and solving for $A_0^*(s)$ and then in steady-state the availability of the system is

$$AV_0(s) = \lim_{s\to 0} s\ AV_0^*(s) = \frac{N_1}{D_1} \qquad \text{…(32)}$$

where

$$N_1 = (\mu_0 + \mu_1)(1 - p_{32}^{(4)}) - \left(\frac{\mu_0 p_{31} - p_{32}^{(4)}}{\lambda}\right) p_{12} \qquad \text{…(33)}$$

and

$$D_1 = (\mu_0\, p_{10} + \mu_1)\, p_{31} + (\mu_0 + \mu_1)\ p_{30}^{(4)} + (\mu_2 + k) p_{12} \qquad \text{…(34)}$$

BUSY-PERIOD ANALYSIS FOR THE EXPERT REPAIRMAN

$$B_0(t) = q_{01}(t) © B_1(t)$$
$$B_1(t) = W_1(t) + q_{10}(t) © B_0(t) + q_{12}(t) © B_2(t)$$
$$B_2(t) = W_2(t) + q_{23}(t) © B_3(t)$$
$$B_3(t) = W_3(t) + q_{31}(t) © B_1(t) + q_{30}^{(4)}(t) © B_0(t) + q_{32}^{(4)}(t) © B_2(t) \qquad \text{…(35-38)}$$

where

$$W_1(t) = e^{-\lambda t}\ \overline{G}(t) \quad ; \quad W_2(t) = \overline{I}(t)$$
$$W_3(t) = e^{-\mu t}\ \overline{G}(t) + [\mu\, e^{-\mu t} © e^{-\lambda t}]\, \overline{G}(t)$$

Taking L.T. and solving for $B_0^*(s)$ and then in steady state, the fraction of time for which the system is under repair is given by

$$B_0 = \lim_{s\to 0} s B_0^*(s) = \frac{N_2}{D_1} \qquad \text{…(39)}$$

where

$$N_2 = \mu_1 (1 - p_{32}^{(4)}) + (\mu_2 + k)\, p_{12} \qquad \text{...(40)}$$

and D_1 is already specified.

EXPECTED NUMBER OF VISITS BY THE EXPERT REPAIRMAN

$$V_0(t) = Q_{01}(t) \text{ (s) } [1 + V_1(t)]$$
$$V_1(t) = Q_{10}(t) \text{ (s) } V_0(t) + Q_{12}(t) \text{ (s) } V_2(t)$$
$$V_2(t) = Q_{23}(t) \text{ (s) (s) } V_3(t)$$
$$V_3(t) = Q_{31}(t) \text{ (s) } V_1(t) + Q_{32}^{(4)}(t) \text{ (s) } V_2(t) + Q_{30}^{(4)}(t) \text{ (s) } V_0(t) \qquad \text{...(41-44)}$$

Taking L.S.T. of these relations and solving for $V_0^{**}(s)$ and then in steady-state the number of visits per unit time is given by

$$V_0 = \lim_{s \to 0} s\, V_0^{**}(s) = \frac{N_3}{D_1} \qquad \text{...(45)}$$

where

$$N_3 = p_{30}^{(4)} + p_{10}\, p_{31} \qquad \text{...(46)}$$

PROFIT ANALYSIS

The expected total profit in steady-state is

$$P_1 = K_0 A V_0 - K_1 B_0 - K_2 V_0 \qquad \text{...(47)}$$

where

K_0 = the revenue per unit up time of the system.

K_1 = is the cost per unit time for which the expert repairman is busy.

K_2 = is the cost per visit for expert repairman.

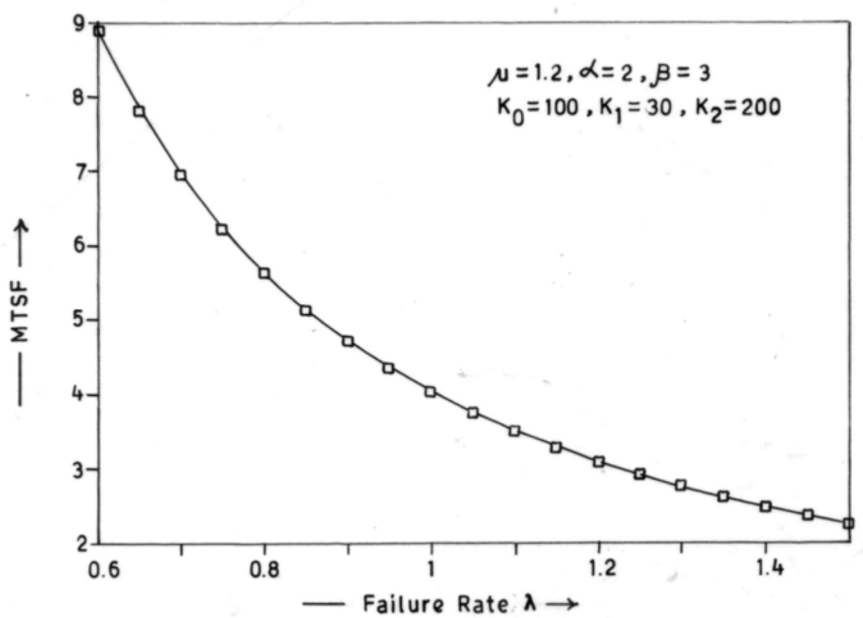


Fig. 2. Behaviour of MTSF w.r.t. failure rate

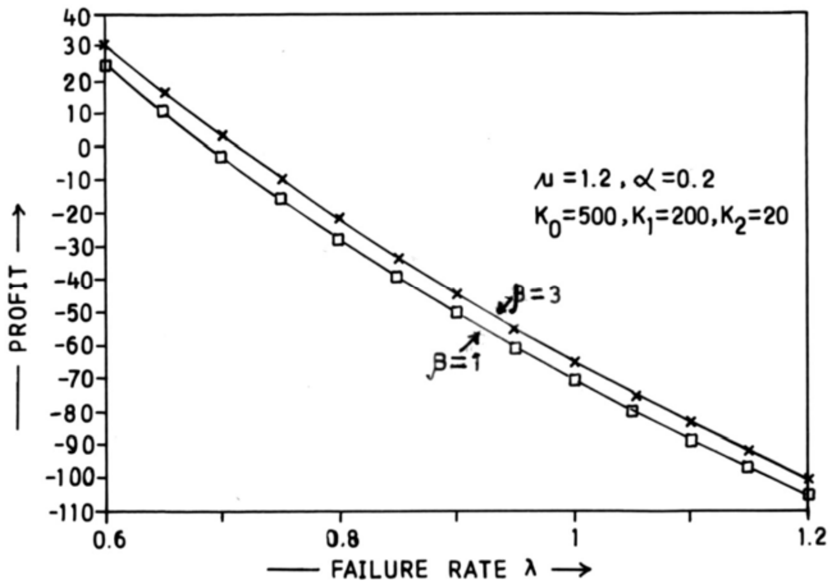


Fig. 3. Behaviour of profit w.r.t. failure rate (λ)

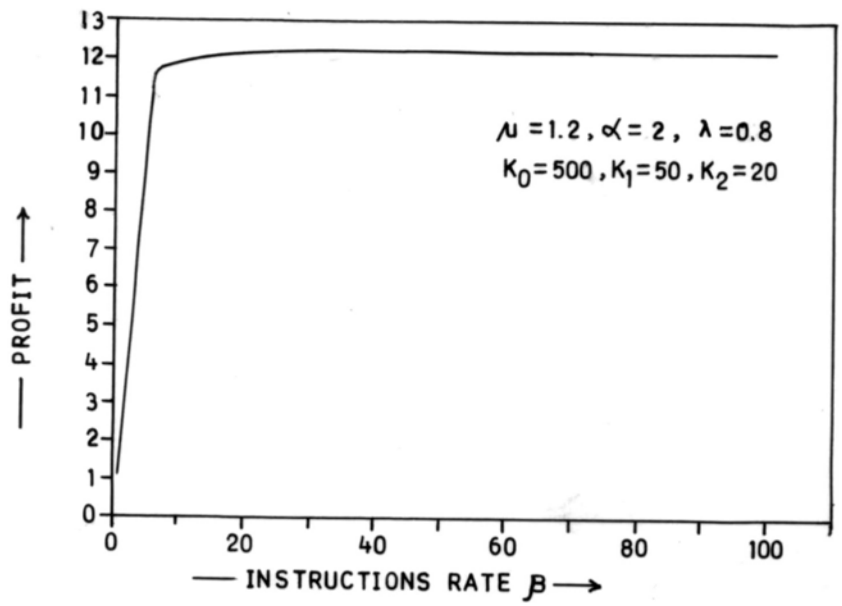


Fig. 4. Behaviour of MTSF w.r.t. instructions rate β.

STUDY THROUGH GRAPHS

For graphical study the following particular case if considered. Let us suppose that $g(t) = \alpha \exp. (-\alpha t)$, $i(t) = \beta \exp. (-\beta t)$ and the remaining distributions are same as taken already. For this case we have plotted the graphs which show :

(i) the behaviour of MTSF with respect to failure rate (keeping other parameters as fixed) as in Fig. 2.

(ii) The behaviour of the expected total profit in the steady-state with respect to failure rate for different values of instructions rate (keeping other parameters as fixed) as in Fig. 3.

(iii) The behaviour of the expected total profit in the steady-state with respect to instructions rate (keeping other parameters as fixed) as in Fig. 4.

It is obvious from Figs. 2 and 3 that MTSF as well as profit decreases as failure rate (λ) increases while profit increases with the increase in instructions rate (β). Also from Fig. 4, we interpret that though the increase in profit is substantial for initial increases in β i.e. profit increases more rapidly up to $\beta = 6$ (nearly) but for higher values profit remains almost constant.

ANOTHER MODEL

We now consider a model wherein the instructions are not given to the assistant repairman. The transition diagram showing the various rates of transition is given in Fig. 5. The various characteristics are obtained as follows :

TRANSITION PROBABILITIES AND MEAN SOJOURN TIMES :

The transition probabilities are :

$$dQ_{01}(t) = \lambda e^{-\lambda_t} dt$$

$$d_{10}(t) = e^{-\lambda_t} g(t)dt$$

$$dQ_{12}(t) = \lambda e^{-\lambda_t} \overline{G}(t)\, dt$$

$$dQ_{10.\underline{23}}(t) = \sum_{n=0}^{\infty} [\lambda e^{-\lambda t} \copyright\ \mu_1 e^{-\mu_1 t} \copyright \{\lambda e^{-\lambda t} \copyright\ \mu_1 e^{-\mu_1 t}\}^{\copyright^n} \copyright\ e^{-\lambda t}]\, g(t)\, dt$$

$$= \left[\frac{\mu_1}{(\lambda+\mu_1)} + \frac{\lambda e^{-(\lambda+\mu_1)t}}{(\lambda+\mu_1)} - e^{-\lambda t}\right] g(t)dt$$

$$dQ_{11.\underline{232}}(t) = \sum_{n=0}^{\infty} [\lambda e^{-\lambda t}\ \copyright \{\mu_1 e^{-\mu_1 t}\ \copyright\ \lambda e^{-\lambda_t}\}^{\copyright^n} \copyright\ e^{-\mu_1 t}\}\, g(t)\, dt$$

$$= \frac{\lambda}{(\lambda+\mu_1)}[1 - e^{(\lambda+\mu_1)t}]g(t)dt \qquad \ldots(48\text{-}52)$$

The non zero element p_{ij} are given below :

$$p_{01} = 1, \quad p_{10} = g^*(\lambda), \quad p_{12} = 1 - g^*(\lambda),$$

$$p_{10.\underline{23}} = \left[\frac{\mu_1 g^*(0)}{(\lambda+\mu_1)} + \frac{\lambda g^*(\lambda+\mu_1)}{(\lambda+\mu_1)} - g^*(\lambda)\right]$$

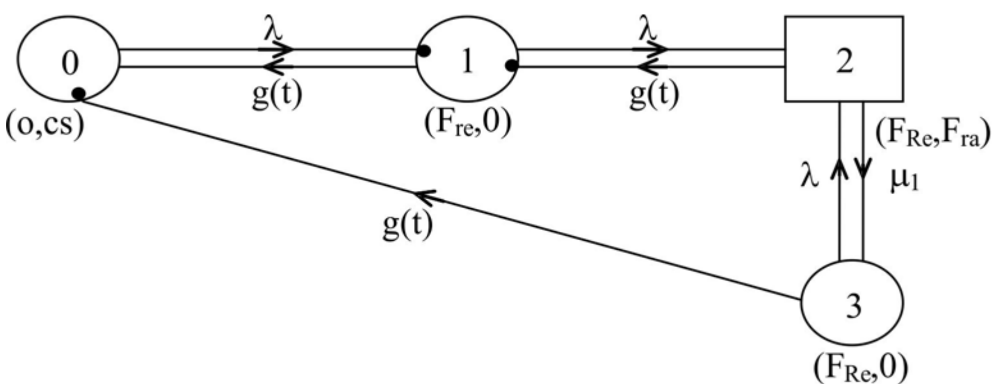
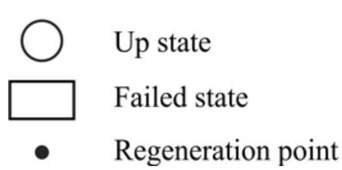


Fig. 5 : State transition Diagram for Model – 2



$p_{11.\underline{232}} = \lambda[g^*(0) + g^*(\lambda + \mu_1)]/(\lambda + \mu_1)$ …(53-57)

By these transition probabilities, it can be verified that

$p_{10} + p_{10,22} + p_{11,232} = p_{10} + p_{12} = 1$ …(58)

Mean sojourn times are :

$\mu_0 = 1/\lambda$, $\mu_1 = (1-g^*(\lambda))/\lambda$,

$m_{21} = 1/\lambda$,

$m_{10} + m_{10.\underline{23}} + m_{11.\underline{232}} = \int_0^\infty td(Q_{10}(t) + Q_{10.\underline{23}}(t) + Q_{11.\underline{232}}(t)] = \in_1$ (say) …(59-62)

MEAN TIME TO SYSTEM FAILURE

The expression for MTSF will remain same as explained in model-1.

AVAILABILITY ANALYSIS

$AV_0(t) = M_0(t) + q_{01}(t) © AV_1(t)$

$AV_1(t) = M_1(t) + [q_{10}(t) + q_{10.\underline{23}}(t)] © AV_0(t) + q_{11.\underline{232}}(t) © AV_1(t)$ …(63-64)

Where

$M_0(t) = e^{-\lambda_1 t}$

$M_1(t) = e^{-\lambda t}\bar{G}(t) + \sum_{n=0}^{\infty}[\lambda e^{-\lambda t} © \mu_1 e^{-\mu_1 t} © \{\lambda e^{-\lambda t} © \mu_1 e^{-\mu_1 t}\}^{©^n} © e^{-\lambda t}]\bar{G}(t)$

$= \frac{1}{(\lambda + \mu_1)}[\lambda e^{-(\lambda+\mu_1)t} + \mu_1]\bar{G}(t)$ …(65-66)

Taking Laplace–transform of equations (65–66) and then letting $s \to 0$, we get

$M_0^*(0) = \mu_0$

$M_1^*(0) = [\lambda(1 - g^*(\lambda + \mu_1)) - \mu_1(\lambda + \mu_1)g^{*\prime}(0)]/(\lambda + \mu_1)^2 = m_1$ (say) …(67-68)

Taking Laplace–transform of equations (63-64) and solving for $AV_0^*(s)$, we obtain

$AV_0^*(s) = N_1(s)/D_1(s)$ …(69)

Where

$N_1(s) = M_0^*(s)(1 - q_{11.\underline{232}}^*(s)) + M_1^*(s)q_{01}^*(s)$

and

$D_1(s) = 1 - q_{11.\underline{232}}^*(s) - q_{01}^*(s)(q_{10}^*(s) + q_{10.\underline{23}}^*(s))$ …(70-71)

The steady–state availability of the system is given by

$AV_0 = \lim_{s\to 0}\left[s\,AV_0^*(s)\right] = N_1/D_1$ …(72)

Where

$N_1 = \mu_0(1 - p_{11.\underline{232}}) + m_1$

and

$D_1 = \in_1 + \mu_0(p_{10} + p_{10.\underline{23}})$ …(73-74)

BUSY–PERIOD ANALYSIS FOR THE EXPERT REPAIRMAN

$B_0(t) = q_{01}(t) © B_1(t)$

$B_1(t) = W_1(t) + \{q_{10}(t) + q_{10.\underline{23}}(t)\} © B_0(t) + q_{11.\underline{232}}(t) © B_1(t)$ …(75-76)

Where

$$W_1(t) = e^{-\lambda t}\bar{G}(t) + \sum_{n=0}^{\infty}[\lambda e^{-\lambda t} © \{\mu_1 e^{-\mu_1 t} © \lambda e^{-\lambda t}\}^{©^n} © e^{-\mu_1 t}]\ \bar{G}(t)$$

$$+ \sum_{n=0}^{\infty}[\lambda e^{-\lambda t} © \mu_1 e^{-\mu_1 t} © \{\lambda e^{-\lambda t} © \mu_1 e^{-\mu_1 t}\}^{©^n} © e^{-\lambda t}]\bar{G}(t)$$

$$= e^{-\lambda t}\bar{G}(t) + \frac{\lambda}{\lambda+\mu_1}[1-e^{(\lambda+\mu_1)t}]\bar{G}(t) + \left[\frac{\mu_1}{(\lambda+\mu_1)} + \frac{\lambda e^{-(1+\mu_1)t}}{(\lambda+\mu_1)} - e^{-\lambda t}\right]\bar{G}(t)$$

$$= \bar{G}(t) \qquad \ldots(77)$$

Taking L.T. of equations (77) and we have

$$W_1^*(0) = -g^{*\prime}(0) = w_1(\text{say})$$

Solving the equations (75-76) with the L.T., we have

$$B_0^*(s) = N_2(s)/D_1(s) \qquad \ldots(78)$$

Where

$$N_2(s) = W_1^*(s)q_{01}^*(s) \qquad \ldots(79)$$

and $D_1(s)$ is already specified.

In steady–state the total fraction of time for which the expert repairman is busy is given by

$$B_0 = \lim_{s\to 0} sB_0^*(s) = N_2/D_1 \qquad \ldots(80)$$

Where

$$N_2 = w_1 \qquad \ldots(81)$$

and D_1 is already specified.

EXPECTED NUMBER OF VISITS BY THE EXPERT REPAIRMAN

$$V_0(t) = Q_{01}(t) ⓢ [1 + V_1(t)]$$

$$V_1(t) = \{Q_{10}(t)\ Q_{10.\underline{23}}(t)\} ⓢ V_0(t) + Q_{11.\underline{232}}(t) ⓢ V_1(t) \qquad \ldots(82\text{-}83)$$

Taking Laplace–Stieltjes transform of these relations and solving $V_0^{**}(s)$, we obtain

$$V_0^{**}(s) = N_3(s)/D_1(s) \qquad \ldots(84)$$

Where

$$N_3(s) = Q_{01}^{**}(s)(1-Q_{11.\underline{232}}^{**}(s))$$

$$D_2(s) = 1.Q_{11.\underline{232}}^{**}(s) - Q_{01}^{**}(s)(Q_{10}^{**}(s) + Q_{10.\underline{23}}^{**}(s)) \qquad \ldots(85\text{-}86)$$

In the steady–state the number of visits per unit time is given by

$$V_0 = \lim_{s\to 0} sV_0^{**}(s) = N_3/D_2 \qquad \ldots(87)$$

Where D_2 is the same as in (74) and

$$N_3 = 1 - p_{11.\underline{232}} \qquad \ldots(88)$$

PROFIT ANALYSIS

The expected total profit in steady–state is

$$P_2 = K_0AV_0 - K_1B_0 - K_2V_0 \qquad \ldots(89)$$

Where

K_0 = revenue per unit up time of the system

K_1 = cost per unit time for which the system is under repair

K_2 = cost per visits by the repairman.

PARTICULAR CASE

Assume that repair time of expert, repairman and instruction time are exponentially distributed :

$g(t) = \alpha e^{-\alpha t}, \; i(t) = \beta e^{-\beta t}$

Then, we have

$p_{01} = 1, \; p_{10} = \alpha/(\alpha+\lambda)$

$p_{12} = \lambda/(\alpha+\lambda)$

$p_{10.\underline{23}} = \lambda\mu_1/(\alpha+\lambda)(\alpha+\lambda+\mu_1)$

$p_{11.\underline{232}} = \lambda/(\alpha+\lambda+\mu_1)$

And mean sojourn times are :

$\mu_0 = 1/\lambda, \; \mu_1 = 1/(\alpha+\lambda)$

$\in_1 = [\alpha(2\lambda+3\alpha)+\lambda^2]/\alpha(\alpha+\lambda)^2$

The results of (72), (80) and (87) are as follows :

Where

$AV_0 = N_1/D_1$ …(90)

Where

$N_1 = (\alpha + \mu_1)(\alpha + \lambda)^3$

and $D_1 = (\alpha + \lambda)(\alpha + \lambda + \mu_1)[\lambda(\alpha + \lambda) + \alpha^2] + \alpha\lambda[\mu_1(\alpha + \lambda) + 2\alpha(\alpha + \lambda + \mu_1)]$

$B_0 = N_2/D_1$ …(91)

Where

$N_2 = \lambda(\alpha + \lambda)^2(\alpha + \lambda + \mu_1)$

$V_0 = N_3/D_1$ …(92)

Where

$N_3 = \alpha\lambda(\alpha + \lambda)^2(\alpha + \mu_1)$

COMPARATIVE STUDY OF THE MODELS THROUGH GRAPHS

For comparison of Model–1 and Model–2, we find the %age increase in profit as $\frac{P_1 - P_2}{P_2} \times 100 = P_0$ (say). Graphs have been plotted as shown in Figs. 6, 7 & 8 and following conclusions are drawn :

(i) If repair of assistant repairman increases then P_0 decreases and also it decreases with increase in failure rate.

(ii) If failure rate (λ) increases, P_0 first increases but after certain value of λ, P_0 goes on decreasing while it increases with increase in instructions rate (β).

(iii) If the value of λ or μ be such that $P_0 > 0$. Model –1 is better than Model–2 and if the value of λ or μ be such that $P_0 < 0$, Model–2 is better than Model–1.

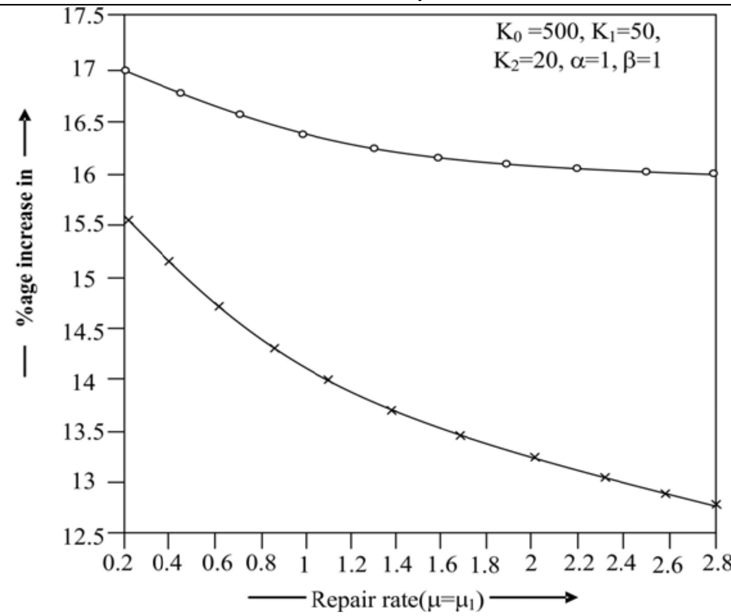


Fig. 6 : %age increase in profit (P0) w.r.t. repair rate (m = m1) for λ = 0.3(o) & 0.6(x).

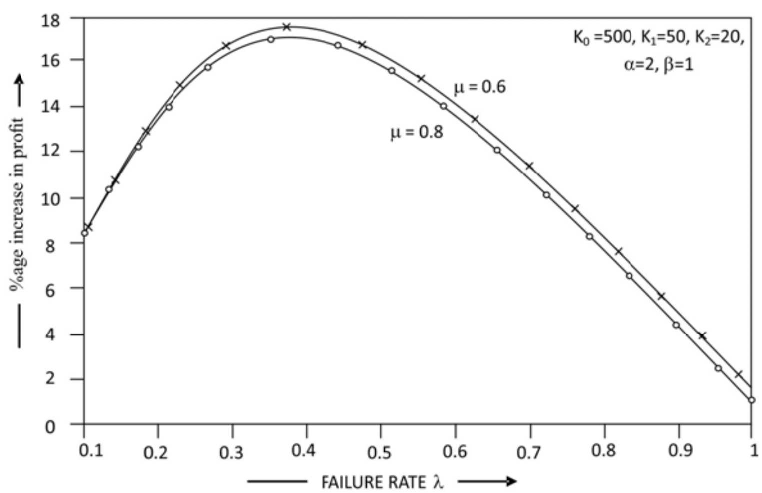


Fig. 7 : %age increase in profit (P_0) w.r.t. failure rate (λ) for μ = 0.6(×) & 0.8(o)

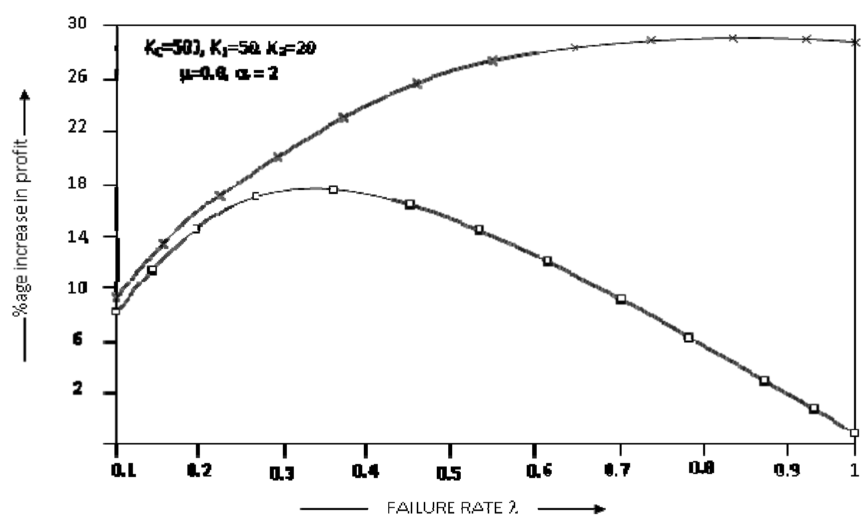


Fig. 8 : %age increase in profit (P_0) w.r.t. failure rate (λ) for β = 1(o) & 6(×)

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CHARACTERIZATION OF INVARIANT STATE

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ABSTRACT :

In this paper we discuss the characterization of invariant state. Definition and examples of various states are also presented. Further, we establish some interesting results with tracial state and B-erogodic state.

INTRODUCTION:

State :

A state on a unital * - algebra A with unit e is continuous linear functional ϕ defined on A satisfying the condition.

$$\|\phi\| = \phi(e) = 1$$

State Space:

The Set S_A with the topology $\sigma(S_A, A)$ is called the state space of A, where S_A is the set of all states on A.

Example (1.1) : Every multiplicative linear functional ϕ defined on a c* algebra is a state if $\|\phi\| = 1$

Example (1.2) :Let $B(H)$ be the class of all bounded operators defined on a Hilbert space H. For a fixed $x \in H$ with $\|X\| = 1$ we define.

$$\phi_X(T) = (Tx, x),$$

where $T \in B(H)$ Than ϕx is state on $B(H)$. Further every state on $B(H)$ is of the form ϕx with $\|X\| = 1$.

Theorem (1.1): An element $x \in A$ is semi positive if and only if $R_e\phi(x) \geq 0$ for every state ϕ on A.

Proof : Let x be a semi-positive element. Then $x + x^*$ is positive. So $\phi(x + x^*) \geq 0$. This gives $\phi(x) + \overline{\phi(x)} \geq 0.$ $ie.$ $R_e\phi(x) \geq 0.$

Conversely, suppose that $R_e\phi(x) \geq 0$ for every state ϕ on A. This means that $\phi(x) + \phi(x^*) \geq 0$ $i.e.$ $\phi(x + x^*) \geq 0.$ Since $x + x^*$ is hermitian, $x + x^*$ is positive. Hence x is semi positive.

Invariant State:

A state ϕ on A is said to be invariant and if $V = [\phi]$ is an invariant sub space of A^*

Now we prove the following characterization of an invariant state.

Theorem (1.2) : A state ϕ on A is an invariant state if and only if $\phi(ax) = \phi(a)\phi(x) = \phi(xa)$ for all a, $x \in A$.

Proof: Suppose that ϕ is an invariant state on A. This means that $V = [\phi]$ is an invariant sub space of A^*. Since

$\phi \in A, \phi \in V$, $L_a \phi$ and $R_a \phi$ are elements of V for all $a \in A$.

Thus $$(L_a \phi)(x) = \phi(ax) = \alpha\phi(x) \qquad (i)$$

and $$(R_a \phi)(x) = \phi(xa) = \beta\phi(x) \qquad (ii)$$

For some scalars α and β where $x \in A$.

Putting $x = e$ in (i) and (ii) we get

$$\alpha = \beta = \phi(a) \qquad (iii)$$

Hence $\phi(ax) = \phi(a)\phi(x) = \phi(xa)$ for all $x, a \in A$.

Conversely, Suppose that $\phi(ax) = \phi(a)\phi(x) = \phi(xa)$

For all $a \in A$ and $x \in A$.

Then

$$(L_a\phi)(x) = \phi(ax) = \phi(a)\phi(x)$$

and $$(R_a\phi)(x) = \phi(xa) = \phi(a)\phi(x)$$

This means that $L_a \phi$, $R_a \phi \in [\phi]$.

Hence $[\phi]$ is an invariant sub space of A^* which proves that ϕ is an invariant state.

Tracial State :

A state ϕ on A is said to be tracial if $\phi(a^* a) = \phi(aa^*)$ for all a $\in A$.

Corollary (I, I) : Every invariant state is a tracial state.

Proof: This follows from the above theorem. B – ergodic state : A state ϕ on A is said to be B – ergodic.

If $$\pi_\phi(A)' \cap \pi_\phi(B)^{11} = [1]$$

For a state ϕ on A let

$$F_{\phi} = \{\psi \in S_A : \psi \leq \lambda\phi \text{ for some } \lambda > 0\}$$

Example (1.3) : Let $A = C^2 + M_2$ where M_2 is the C^* - algebra of 2×2 complex matrices and let β be the C^* - sub algebra spanned by the unit of A and the projection.

$$P = \left(1, 0, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right),$$

we define states ϕ_1, ϕ_2, and ϕ_3 of A as follows :

$$\phi_1\left(\alpha, \beta, \left(\lambda_{ij}\right)\right) = \alpha$$

$$\phi_2\left(\alpha, \beta, \left(\lambda_{ij}\right)\right) = \beta$$

$$\phi_3\left(\alpha, \beta, \left(\lambda_{ij}\right)\right) = 11$$

Let $\psi = \frac{1}{2}(\phi_1 + \phi_2)$ and $\phi = \frac{1}{3}(\phi_1 + \phi_2 + \phi_3)$

Then $\psi \in F_{\phi}$, ϕ B-ergodic, but ψ is not B – egrodic.

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DAMPED VIBRATION OF RECTANGULAR PLATES OF PARABOLICALLY VARYING THICKNESS RESTING ON ELASTIC FOUNDATION CONSIDERING THE EFFECT OF THERMALLY INDUCED NON-HOMOGENEITY

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ABSTRACT :
A mathematical model is developed to assist the engineers by studying and analyzing the effect of damping and elastic foundation on natural frequencies of a non-homogenous (thermally induced) rectangular plate of parabolically varying thickness. Frequencies corresponding to the first three modes of vibration are computed for the rectangular plate with clamped- simply supported -clamped -simply supported (C-SS-C-SS), clamped-simply supported-simply supported-simply supported (C-SS-SS-SS) edge conditions for different values of taper constants, damping parameter and elastic foundation parameter. Results are depicted graphically to comprehend the results in easy way.
Keywords: ***Non homogeneity, damping constant, elastic foundation, taper constant, frequency parameter.***

INTRODUCTION:

The study of vibration has been of principal concern for a very long time. Mathematicians, design engineers and civil engineers have always been fascinated to study the response of vibration on various structures. Nearly all the structures are effected by the vibration since their construction require plate's structure. Controlled plate vibration are of advantage in providing Safety, Durability and Economy. They are also useful in Schedule maintenance repairs and Technological Advancement. Study of Controlled plate vibration also helps us to be Environment friendly and protecting environment. Though effect of vibration can be seen from kitchen (juicer, grinder, etc.) to gym (whole body massager) i.e. in every phase of life but Major areas of application of plate vibration are in the following: Earth quake resistant structures, Design of air craft wing section, Multi storey buildings, Telephone Industries, Marine structure, Nuclear reactor technology, Railways, satellite antennas, space structure etc.

Vibration analysis has its beginning with Galilei (1564-1642) who solved by geometrical means dependence of natural frequency of simple pendulum .After that Mersenne in 1635 recognized that vibration is inversely proportional to length of string and directly proportional to the square root of the cross sectional area. Euler and Bernoulli had also solved various problems on vibration of plates in mid of 18^{th} century but comprehensive work was done in field of vibration of plates by Leissa [8] who presented the detailed literature on it. Various authors like Nagaraja and Rao [9], Stanic[12], Young[14], Das[2], Hamada [6],Sezawa[11], Jain and Soni [7], Shalini etal [13] studied rectangular and square plates with different boundary conditions using various techniques for solution. Recently Gupta M., Kumar and Jain [3] has worked on problem "Damped vibration of exponentially non homogenous isotopic elastic rectangular plates of linearly varying thickness resting on elastic foundation".

Present work is devoted to study the effect of damping and elastic foundation on natural frequencies of a non-homogenous induced thermally in rectangular plate of parabolically varying thickness. Frobenious method has been applied for the solution of the differential equation obtained and Frequencies corresponding to the first three modes of vibration are computed and presented graphically for the rectangular plate with clamped- simply supported-clamped -simply supported (C-SS-C-SS), clamped-simply supported-simply supported -simply supported (C-SS-SS-SS) edge conditions for different values of taper constants, damping parameter and elastic foundation parameter.

EQUATION OF MOTION

The equation of motion of an element of plates in transverse direction are given by equations

$$\nabla^2\left(D\nabla^2 w\right)-(1-\upsilon)\left[\frac{\partial^2 D}{\partial y^2}\frac{\partial^2 w}{\partial x^2}-2\frac{\partial^2 D}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial^2 w}{\partial y^2}\frac{\partial^2 D}{\partial x^2}\right]$$
$$+\rho h\frac{\partial^2 w}{\partial t^2}+K\frac{\partial w}{\partial t}+K_f w=0 \qquad \text{(I)}$$

Where w= deflection function, D=flexural rigidity, υ =Poisson's ratio, ρ= density, K=damping constant, K_f = elastic foundation constant.

Here we assume that the two opposite edges y=0 and y=b of the plate be simply supported and thickness various parabolically along the length i.e. in the direction x-axis. Thus, h is independent of y i.e. h=h(x).For a harmonic solution, the deflection function w satisfying the condition at y=0 and y=b, is

$$w(x,y,t)=W(x)\sin\frac{m\pi y}{b}e^{-\gamma t}\cos pt \qquad \text{(II)}$$

We introduce the non-dimensional variables

$$H=\frac{h}{a},\ \bar{E}=\frac{E}{a},\ x=\frac{X}{a},Y=\frac{y}{a},\bar{\rho}=\frac{\rho}{a},\ W=\frac{\bar{W}}{a},\ s^2=\beta^2a^2$$

And let the thickness of the plate varies parabolically i.e

$$H=H_0\left(1-\alpha X^2\right) \qquad \text{(III A)}$$

Also we let

$$E=E_0\left(1-\eta(1-X)\right)$$

Where η is thermal gradient and α is taper constant. (IIIB)

Thus we obtain the equation of plate motion as following

$$\left(1-\eta(1-X)\right)^2\left(1-\alpha X^2\right)^4\frac{\partial^4\bar{W}}{\partial X^4}+\left[-12\alpha X\{1-\eta(1-X)\}^2\left(1-\alpha X^2\right)^3+2\eta\{1-\eta(1-X)\}\left(1-\alpha X^2\right)^4\right]\frac{\partial^3\bar{W}}{\partial X^3}$$
$$+\left\{\begin{array}{l}-6\alpha\left(1-\eta(1-X)\right)^2\left(1-\alpha X^2\right)^3-12\alpha\eta X\left(1-\eta(1-X)\right)\left(1-\alpha X^2\right)^3+\\ 24\alpha^2X^2\left(1-\eta(1-X)\right)^2\left(1-\alpha X^2\right)^2-2s^2\left(1-\eta(1-X)\right)^2\left(1-\alpha X^2\right)^4\end{array}\right\}\frac{\partial^2\bar{W}}{\partial X^2}$$
$$+\left[12s^2\alpha X\left(1-\eta(1-X)\right)^2\left(1-\alpha X^2\right)^3-2\eta s^2\left(1-\eta(1-X)\right)\left(1-\alpha X^2\right)^4\right]\frac{\partial\bar{W}}{\partial X}+\left\{s^4\left(1-\alpha X^2\right)^4\left(1-\eta(1-X)\right)^2\right\}\bar{W}$$
$$-\nu^2s^2\left[-12\alpha\eta X\left(1-\eta(1-X)\right)\left(1-\alpha X^2\right)^3+24\alpha^2X^2\left(1-\eta(1-X)\right)^2\left(1-\alpha X^2\right)^2-6\alpha\left(1-\eta(1-X)\right)^2\left(1-\alpha X^2\right)^3\right]\bar{W}$$

$$-\left\{D_K{}^2 M^{*2} + \Omega^2 M^*\left(1-\eta\left(1-X\right)\right)^2\left(1-\alpha X^2\right)^2 - \left(1-\eta\left(1-X\right)\right)\left(1-\alpha X^2\right)F_P\big/C^*\right\}\bar{W} = 0 \tag{IV}$$

Here Ω is frequency parameter, D_K is damping parameter and F_P is foundation parameter.

SOLUTION OF EQUATION

A series solution for $\bar{W}$ is then assumed to be in the form,

$$\bar{W}\left(X\right) = \sum_{\lambda=0}^{\infty} a_\lambda X^{c+\lambda}, a_0 \neq 0 \tag{V}$$

For the series expression to be the solution, the coefficients of the powers of X in the equation must be identically zero. Thus by equating the coefficients of the lowest power of X to zero, the following indicial roots are obtained:C=0, 1, 2, and 3. On equating the coefficients of the higher powers of X to be zero, it is found that the constants a_1, a_2 and a_3 are indeterminate for C=0, so these can be taken as arbitrary constants along with a_0. The remaining constants $\mathbf{a}_\lambda$ (λ= 4, 5, 6………..) are all obtained in terms of a_0, a_1, a_2 and a_3. The remaining unknown constants are determined from recurrence relation:

$$a_\lambda = a_0 R_\lambda\left(0\right) + a_1 R_\lambda\left(1\right) + a_2 R_\lambda\left(2\right) + a_3 R_\lambda\left(3\right) \tag{VI}$$

Where $R_\lambda\left(i\right) = 1, \lambda = i$

= 0, otherwise for $\lambda = i = 0,1,2,3$

$R_\lambda(0)$, $R_\lambda(1)$, $R_\lambda(2)$, $R_\lambda(3)$ for (λ=3,4,5,6…..) are functions ofα,s η, M*,C*,D_K ,F_P and Ω. The solution for $\bar{W}$, corresponding for c=0 is

$$\bar{W} = a_0\left[1+\sum_{\lambda=4}^{\infty} R_\lambda(0) X^\lambda\right] + a_1\left[X+\sum_{\lambda=4}^{\infty} R_\lambda(1) X^\lambda\right] + a_2\left[X^2+\sum_{\lambda=4}^{\infty} R_\lambda(2) X^\lambda\right] + a_3\left[X^3+\sum_{\lambda=4}^{\infty} R_\lambda(3) X^\lambda\right] \tag{VII}$$

It is evident that no new solution will arise corresponding to other values of c i.e. for c=1,2,3. Solutions corresponding to these values of c are already included in the solution corresponding to c=0.

CONVERGENCE OF THE SOLUTION

To test the convergence of the solution the technique used by Lamb has been applied here. From this technique and using recurrence relation one obtains the required equation as,

$$\mu^4\left(\mu - \alpha\right)^4\left[\mu\left(1-\eta\right)+\eta\right]^2 = 0 \tag{VIII}$$

Where

$$\mu = \lim_{\lambda\to\infty}\frac{a_{\lambda+1}}{a_\lambda}$$

We get roots of the equation as

$$\mu = 0,0,0,0,\alpha,\alpha,\alpha,\alpha,\frac{-\eta}{1-\eta},\frac{-\eta}{1-\eta}$$

Hence the solution is convergent for

$$|\alpha| \prec 1, \left|\frac{-\eta}{1-\eta}\right| \prec 1$$

Boundary Conditions and frequency equations

The following boundary conditions have assumed for the present problem:

Clamped Edge Conditions (C-SS-C-SS)

At a clamped edge

$$w = 0, \quad \frac{\partial w}{\partial x} = 0 \tag{IX}$$

Simply Supported Edge Conditions (C-SS-SS-SS)

At a simply supported edge

$$w = 0, \quad M_x = 0 \tag{X}$$

These conditions (IX) and (X) are applied on equation (VII) and we obtain the frequency equations as:

$$\begin{vmatrix} V_1 & V_2 \\ V_3 & V_4 \end{vmatrix} = 0 \tag{XI A}$$

$$\begin{vmatrix} V_1 & V_2 \\ V_5 & V_6 \end{vmatrix} = 0 \tag{XI B}$$

RESULTS:

Frequency equations (XI A) and (XI B) provide the value of frequency parameter for first three modes of vibration for different values of taper constant, thermal gradient parameter, damping parameter, elastic foundation parameter for both the plates. In all cases Poisson ratio has been assumed to remain constant and given value 0.3. Though many combination of parameters are possible but here due to space constraints we discussed the eight cases only. All results are displayed graphically for both the boundary condition. The results so obtained compare well with already published work.

Figure 1 depicts Variation of frequency parameter (Ω) for vibration of a homogeneous rectangular plate of parabolically varying thickness for different values of taper constant.

Figure 2 depicts Variation of frequency parameter (Ω) for vibration of a damped homogeneous rectangular plate of parabolically varying thickness for different values of taper constant.

Figure 3 explains Variation of frequency parameter (Ω) for vibration of a damped homogeneous rectangular plate of parabolically varying thickness resting on elastic foundation for different values of taper constant.

Figure 4 explains Variation of frequency parameter (Ω) for vibration of a homogeneous rectangular plate of parabolically varying thickness for different values of damping constant.

Figure 5 displays Variation of frequency parameter (Ω) for vibration of a homogeneous rectangular plate of parabolically varying thickness resting on elastic foundation for different values of damping constant.

Figure 6 displays Variation of frequency parameter (Ω) for vibration of damped homogeneous rectangular plate of parabolically varying thickness for different values of foundation parameter.

Figure 7 depicts Variation of frequency parameter (Ω) for vibration of damped non- homogeneous rectangular plate of parabolically varying thickness for different values of taper constant

Figure 8 displays Variation of frequency parameter (Ω) for vibration of damped non- homogeneous rectangular plate of parabolically varying thickness for different values of thermal gradient.

CONCLUSIONS:

From above results, it is observed that on increasing the value of taper constant there gradual decrease in the value of frequency parameter for the three modes of vibration but major effect is on third mode rather than first and second modes. Also it is observed that on applying damping the values frequency parameter decreases for the three modes but if thermal gradient is also applied to the damped plate effect is reversed. The value of frequency parameter goes on increasing with increase in elastic foundation parameter for all the modes in both the boundary conditions but here again the effect is more on first and third modes of vibrations. The effect of thermal gradient shows that frequency parameter for third mode increases sharply as its value increases but for second mode it remains almost constant. We also conclude that frequency parameter increases for first mode when there is increase in thermal gradient parameter but the increase not as sharp as third mode.

Fig 1: Variation of Ω for vibration of a homogeneous rectangular plate of parabolically varying thickness for different values of taper constant
H_0 = 0.03, υ = 0.3, m= 1.0, a/b = 0.25, D_k= 0.0, F_P=0.0, η=0.0

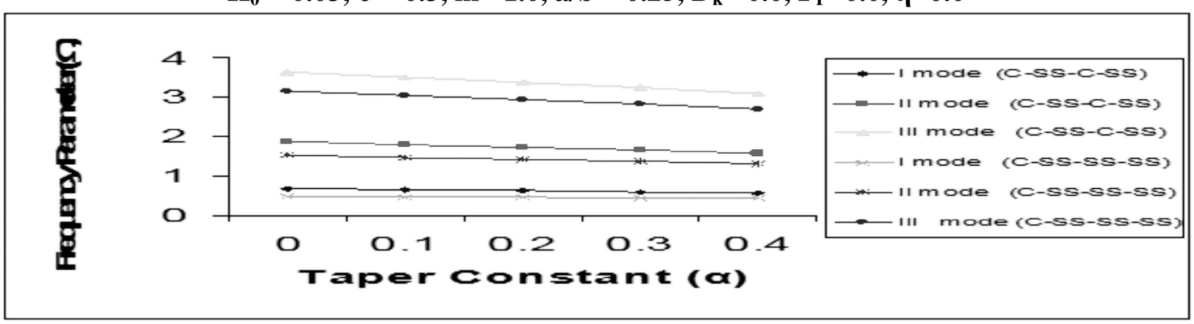


Fig 2: Variation of Ω for vibration of a damped homogeneous rectangular plate of parabolically varying thickness for different values of taper constant
H0 = 0.03, υ = 0.3, m= 1.0, a/b = 0.25, Dk= 0.0025, FP=0.0, η=0.0

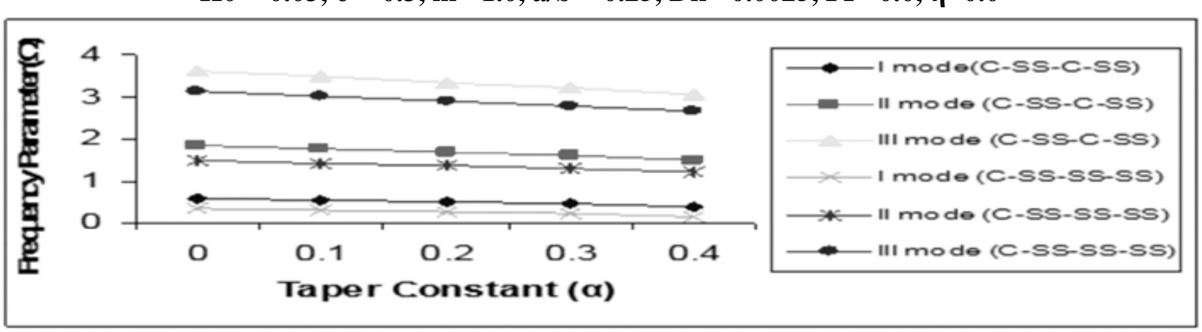


Fig 3: Variation of Ω for vibration of a damped homogeneous rectangular plate of parabolically varying thickness resting on elastic foundation for different values of taper constant
$H_0 = 0.03$, $\upsilon = 0.3$, m= 1.0, a/b = 0.25, D_k= 0.0, F_P=0.01, η=0.0

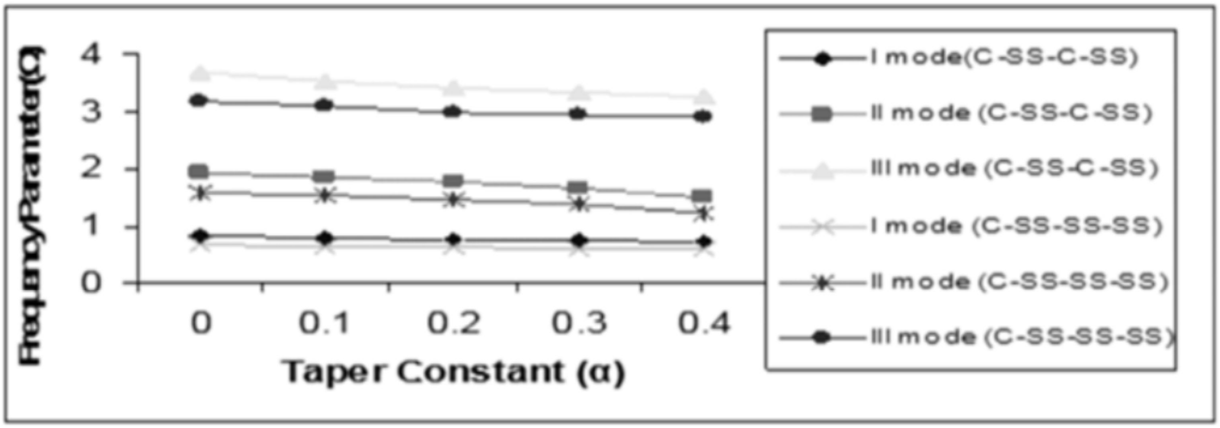


Fig 4: Variation of Ω for vibration of a homogeneous rectangular plate of parabolically varying thickness for different values of damping constant.
$H_0 = 0.03$, $\upsilon = 0.3$, m= 1.0, a/b = 0.25, F_P= 0.0, α=0.4, η=0.0

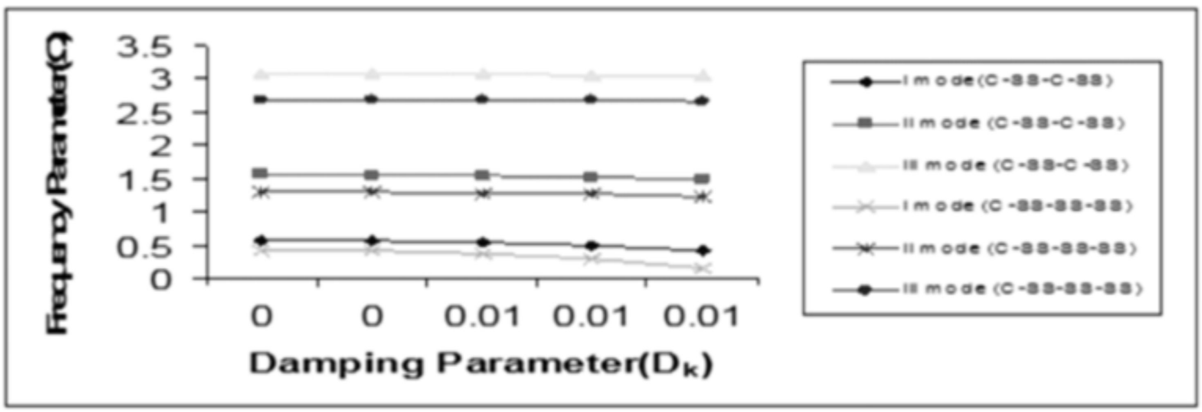


Fig. 5: Variation of Ω for vibration of a homogeneous rectangular plate of parabolically varying thickness resting on elastic foundation for different values of damping constant.
$H_0 = 0.03$, $\upsilon = 0.3$, m= 1.0, a/b = 0.25, F_P= 0.01, α=0.4, η=0.0

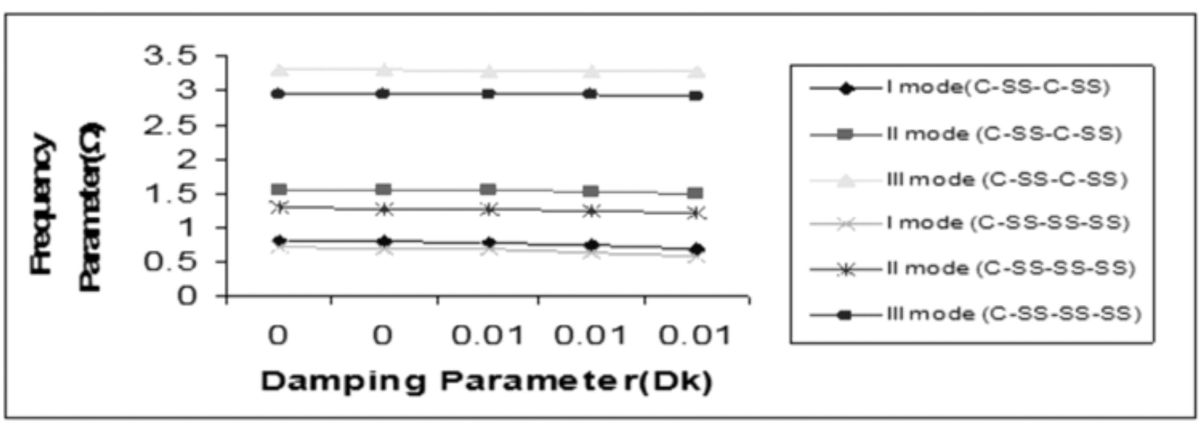


Fig. 6: Variation of Ω for vibration of damped homogeneous rectangular plate of parabolically varying thickness for different values of foundation parameter.
H_0 = 0.03, υ = 0.3, m= 1.0, a/b = 0.25, D_k= 0.01, α=0.4,η=0.0

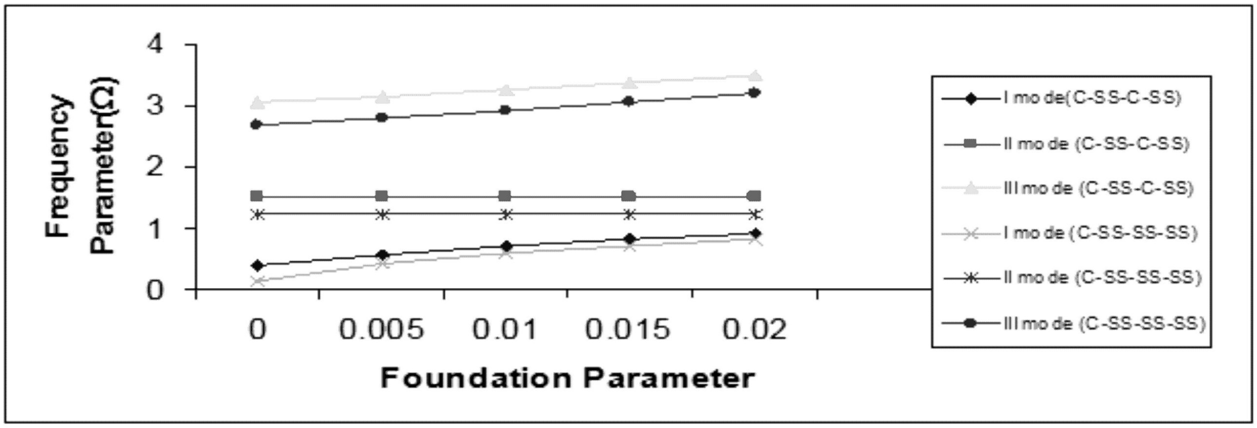


Fig 7: Variation of Ω for vibration of damped non- homogeneous rectangular plate of parabolically varying thickness for different values of taper constant
H_0 = 0.03, υ = 0.3, m= 1.0, a/b = 0.25, D_k= 0.01, α=0.4, η=0.1, F_P=0.01

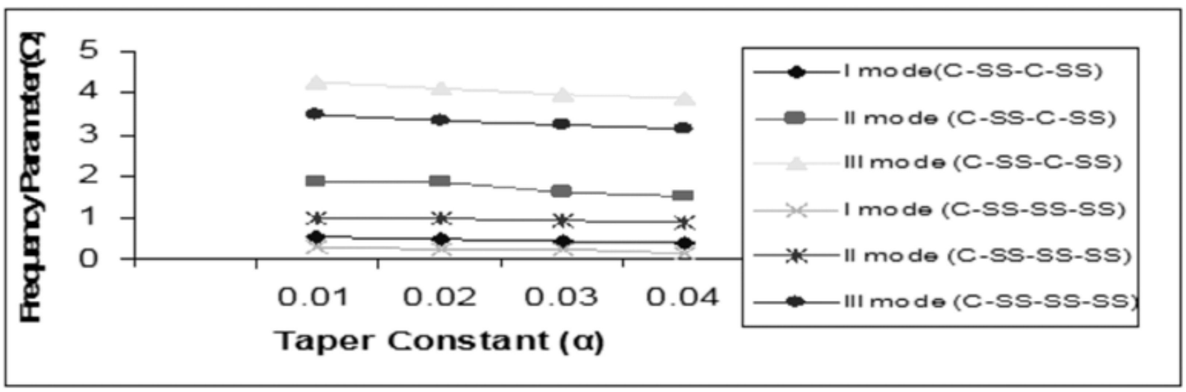
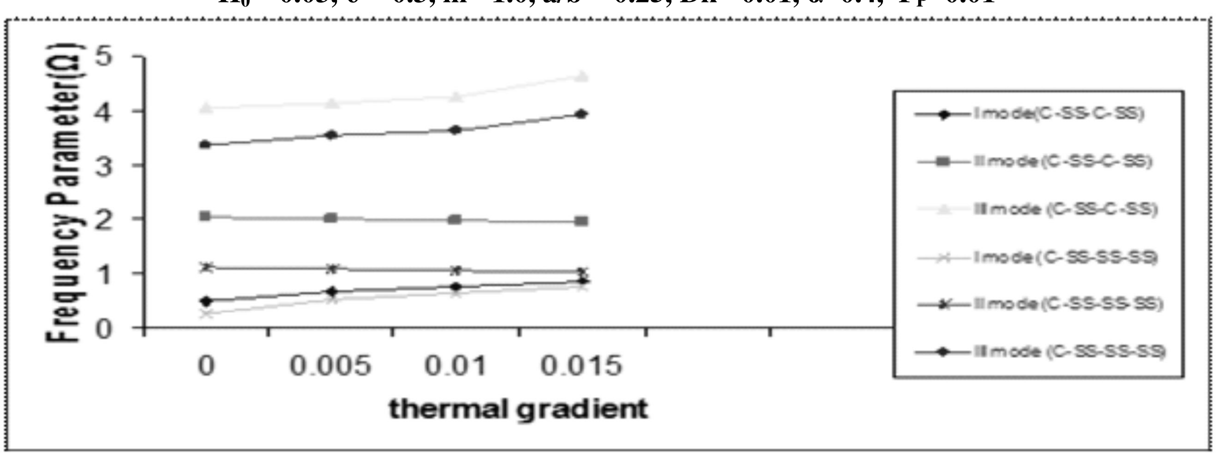


Fig 8: Variation of Ω for vibration of damped non- homogeneous rectangular plate of parabolically varying thickness for different values of thermal gradient
H_0 = 0.03, υ = 0.3, m= 1.0, a/b = 0.25, Dk= 0.01, α=0.4, F_P=0.01



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MULTI-LEVEL MODELING APPROACH IN ANALYZING THE ASSOCIATION BETWEEN UTILIZATION OF ANTENATAL CARE SERVICES AND EXPLANATORY VARIABLES

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ABSTRACT :

One of the goals of reproductive and child health programme in India is to reduce the infant and maternal morbidity in the society. To achieve these objectives, the RCH service package has been introduced under the programme in all the states of India. This can be achieved by maximizing the utilization of these services. The goal of maximizing the utilization could be achieved by identifying the factors responsible for utilization of services. There have been many studies in the past which finds the association of the RCH service utilization and various factors influencing the utilization, but there are very few studies, which consider the use of multilevel approach of regression. The present study aims at emphasizing the role of hierarchical modeling in analyzing the association between the complete ANC services utilization, one of the components of RCH services package, with the individual level, Community level and District level factor.

Keywords : ***RCH, ANC, PNC, multilevel modeling, DLHS***

INTRODUCTION

The overall goal of reproductive and child health program in the country is to reduce the infant and maternal mortality and morbidity and increase contraceptive prevalence rate in the society. To achieve these objectives, the RCH service package, with the component of Maternal Health, Child Health, Family Planning, Adolescent Reproductive and Sexual Health, Urban Health, Tribal Health, has been introduced under the program in all the states of India.This goal can only be achieved by maximizing the utilization of these services and by utilizing the factors associated with the same.

The present study aims at emphasizing the role of hierarchical modeling in analyzing the association between the utilization of complete ANC services with the individual level, Community level and District level factor.

The specific objectives of the study are:

a. To assess the inequity in the access and utilization of different RCH services, viz, maternal (ANC and PNC).
b. To investigate the degree to which the ANC utilization is influenced by the contexts within which the people live and other explanatory variables and to assess the superiority of Multi-level modeling approach over the Standard Logistic Regression for the current situation.

MATERIAL AND METHODS

The information collected by District Level Household &Facility Survey (DLHS-RCH III: 2005-06)[1] round three survey for the state of Madhya Pradesh is used to examine the access/utilization of the health programs.

The outcome indicators and the explanatory variables are given in the annexure-A.

Univariate analysis was carried out to filter out the insignificant determinants for further analysis.

For testing the co-linearity of the explanatory variables we have analysed the magnitude of multi-colinearity by considering the size of the Variance inflation factor VIF(β_i). A common rule of thumb is that if VIF(β_i) >5 then multi-colinearity is high. After checking the VIF of all the explanatory variables, binary logistic regression with step wise method was used.

The multilevel modelling approach[2] was also adopted for analysing the dependent nature of the outcome variable. Comparison of the two approaches is also done to evaluate the superiority of the one over the other.

The model at **level-1** (The Traditional Logistic Regression model) can be formally expressed as:

$$y_{ij}=\beta_{0j}+\sum_{l=1}\beta_{lj}\chi_{lj}+e_{ij}$$

where χ_{ij} *'s are explanatory variables*

β_{0j} *is the value of* $E(Y_{ij})$ *when* $\chi_{ij}=0$ *for all* $i=1,2,\ldots,p$

e_{ij} = *residual* variation in y_{ij} that cannot be explained.

$$e_{ij}\overset{iid}{\sim} N(0, \sim\sigma_e^2)$$

To make this a **two-level** random intercept model, we let β_{0j} become a random variable, with an assumption that:

$\beta_{0j} = \beta_0+u_{0j}$, where u_{0j} *is the random part*

A multilevel model based on a **three-level** structure of individuals (level-1 denoted by 'i' subscript) nested within neighbourhoods/Primary Sampling Units (PSUs) (level-2, denoted by 'j' subscript) nested within regions/district (level-3, denoted by 'k' subscript). The micro model can be written as:

$$y_{ijk} = \beta_{0jk}+\beta_1 x_{1ijk}+e_{0ijk}$$

$$y_{ijk}=\beta_{0jk}+\sum_{l=1}\beta_{ljk}\chi_{ljk}+e_{ijk}$$

with an additional subscript to represent the regions. In addition, there would be a macro model at the neighbourhood level (level-2):

$$\beta_{0jk}=\beta_{0k}+u_{0jk}$$

where, u_{0jk} is the differential for the *jth* neighborhood in the region from average β_{0k}. There would also be a macro model at the region level (level-3):

$$\beta_{0k}=\beta_0+u_{0k}$$

where, υ_{0k} is the differential for the *k*th region from the average β_0

Analytical Framework

Inequity in access and utilization of under consideration RCH services would be determined by seeing the access utilization differentials.

As one of the key aims of this study is to investigate the degree to which the RCH utilization is influenced by the contexts within which the people live, the use of a conventional regression analysis framework has critical limitations. Multilevel statistical techniques provide a technically robust framework, to analyze outcome variable, overcoming the limitations of traditional regression technique[5].

RESULTS

ANC and treatment seeking behaviour for pregnancy problems

It was observed that the registration for ANC was done by 51% of the pregnant women and only 6.3% of the women received the full antenatal care, i.e. 3 ANC visits, 2 TT and at least 90 IFA tablets or equivalent amount of iron syrup. Only 46% of the women sought treatment for their pregnancy related problems. Very few women were getting facilitation or motivation for availing antenatal care. Only 43% of pregnant women opted for safe delivery, i.e, going to institution, conducting delivery by doctor or trained personnel. It was also noticed that only 31% went for check up within 48 hours after the delivery.

Outcome indicator by background characteristics of the mother

The health seeking behavior, viz., registration of pregnancy for ANC, Full ANC, treatment sought for pregnancy related problems safe delivery and PNC with 48 hours of delivery, were in higher proportion in those villages which had primary or middle school in the village, had government health facility, availability of health provider in the village or distance to nearest health facility was less than 5 kms.All the indicators of ANC and PNC care were positively affected by the household level indicators and also by the individual level indicator.

Association of Complete ANC with the Contextual and Individual Level Factors: Traditional Logistic Regression Vs Multi-Level Modeling Approach

In this section we explored the association of the maternal health indicators, viz, Complete ANC with the covariates at individual and community level. Also, the traditional logistic regression and multilevel modelling approach was compared to see the interplay of outcome and independent variables under the two different approaches [9].

The traditional regression logistic regression analysis (Table 1) shows that there is statistically significant association between the complete ANC and the household level covariates viz, type of drinking water used in the household, type of toilet in the household and individual level covariates viz, facilitation for availing ANC by the doctor, heard or seen message regarding the ANC care, education of the women and also the number of children ever born to the women.

Table 1 : Association of Complete ANC with covariates at individual and community level - Traditional Logistic Regression Analysis

Covariates (Reference Category)	Odds Ratio	CI for OR	
Water Treatment (Treated)	.73	.62	.85
Type Of Toilet (flush)	.88	.78	.99
Faciliatation Or Motivation For Availing Antenatal Care-Doctor (Yes)	.78	.62	.97
Heard/Seen Message-Antenatal Care (Yes)	.34	.20	.59
Women Education (Illiterate)	1.33	1.23	1.44
Total Number Of Children	.89	.81	.97

The two level model with district at the second level (Table 2) shows that there is statistically significant association between the complete ANC and the household level covariates viz, type of drinking water used in the household and individual level covariates viz, facilitation for availing ANC by the doctor, heard or seen message regarding the ANC care, education of the women and also the number of children ever born to the women.

LR test for multilevel VS the traditional regression model results in a Chi-square statistics value 38.37 with p-value<0.001 and shows a strong advocacy for the two level model.

Table 2 : Association of Complete ANC with covariates at individual and community level - Multilevel Logistic Regression Analysis (MLR-1)

Full_ANC	Odds Ratio	CI for OR	
Water Treatment	.81	.68	.96
Facilitation or Motivation for Availing Antenatal Care by Doctor	.78	.62	.97
Heard/Seen Message-Antenatal Care	.37	.21	.65
Women education	1.31	1.21	1.43
Total Number Of Children	.89	.82	.97

Std. Dev. (cons)= .3812742 with CI (.2728894 .5327067)

LR test vs. logistic regression gives χ^2=38.37 with p=<0.001

The 3 level model (Table 3) shows that there is statistically significant association between the complete ANC and the household level covariates viz, type of drinking water used in the household and individual level covariates viz, facilitation for availing ANC by the doctor, heard or seen message regarding the ANC care, education of the women and also the number of children ever born to the women.

The LR test for multilevel VS the traditional regression model results in a Chi-square statistics of value 40.84 with p-value<0.001, which is even greater than the 2 level model, shows a stronger advocacy for the three level modeling.

Table 3 : Association of Complete ANC with covariates at individual and community level - Multilevel Logistic Regression Analysis (MLR-2)

	Odds ratio	CI for OR	
Water Treatment	.80	.68	.95
Facilitation or Motivation for Availing Antenatal Care by Doctor	.77	.61	.96
Heard/Seen Message-Antenatal Care	.37	.21	.64
Women education	1.32	1.21	1.44
Total Number of Children	.89	.82	.97

Random-effects	Parameters	Estimate	[95 Conf. Interval]	
District				
	SD(constant)	.38	.27	.54
Village(PSU)				
	SD(constant)	.36	.19	.71

LR test vs. logistic regression: chi2(2) = 40.84 Prob> chi2 = 0.0000

DISCUSSION & CONCLUSION

In 2004, one of the studies by Pandey et al[3], study on Maternal and Health Care Services : Observations from Chattisgarh, Jharkhand and Uttaranchal, examined the pattern and correlates of utilization of antenatal care services and assistance received during delivery in these three recently formed states, which have distinct geographical and topographical characteristics. The study emphasizes the role of geographical as well as socio-economic factors for reaching out to the people.

A study by Jatet al[4], showed very strong positive influence of higher household socio-economic status on the use of ANC, PNC and skill attendance at delivery in the state of Madhya Pradesh. This study found sufficient amount of variation at community and district of residence on each of the three indicators of the use of maternal health services.

Another study by Stephenson et al[5], on influence of community-level characteristics on the use of maternal and reproductive health services conducted in Uttar Pradesh reported strong community- level influence on service use. This study further highlighted that the role of some individual and household factors in determining a person's use of services was mediated by the characteristics of the community in which the individual lives.

Our study suggests, when comparing the traditional regression model and multilevel model with same covariates, we find that the role of some of the individual and household covariates in utilizing of the complete ANC is mediated by the community level covariates. Though none of the community level covariates selected in our study were found to be significant, yet it is evident from the LR test for multilevel Vs the traditional regression model, that the multilevel approach render a better regression model than the traditional regression model.

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Annexture-A : Outcome and Explanatory Variables used in the study

Outcome Indicator	**Explanatory Variables**		
	Level 3 : District Level	**Level 2 : Village level**	**Level 1 : Household and individual level**
Received Complete ANC	District indicator	Primary or Middle School	Own Electricity
Institutional Delivery		Any Govt Health Facility in the Village	Religion
Check up within 48 hours after delivery		Health Provider In The Village	Caste Group
		Drainage Facility-Available	Water Treatment
		Distance To Nearest Bus Station	Type Of Toilet
			Type Of Fuel Used In The Kitchen
			Type Of House
			Separate Room For Kitchen
			Wealth Index Quintiles
			Age Group
			Age at Consummation of Marriage
			Women's education level
			Husband's Education level
			Living With Husband or Husband Staying Elsewhere
			Age at First Birth
			Total Number of Children
			Facilitation or Motivation for Availing Antenatal Care-Doctor
			Facilitation or Motivation for Availing Antenatal Care-ANM
			Facilitation or Motivation for Availing Antenatal Care-Health Workers
			Facilitation or Motivation for Availing Antenatal Care-Anganwadi Worker
			Facilitation or Motivation for Availing Antenatal Care-ASHA
			Heard/Seen Message-Antenatal Care Heard/Seen Message-Institutional Delivery

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AN EXTENSION OF GENERALIZED EXPONENTIAL DISTRIBUTION FOR DETERMINING OVARIAN RESERVE AND REPRODUCTIVE AGE FROM MEASUREMENT OF OVARIAN VOLUME

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ABSTRACT :
The two parameters generalized exponential distribution has been used to analyse lifetime data. In this paper the two parameter generalized exponential distribution has been embedded in a larger class of distributions obtained by introducing another shape parameter. Because of the additional shape parameter more flexibility has been introduced in the family. Here we can clearly observe proportional reversed hazard family of distributions which can be used to analyse data. In our application we have considered the analysis of data based on ovarian reserve and reproductive age. The measurement of the ovarian volume is compared with different reproductive ages which is directly associated with the primordial follicle population.
Keywords : ***Generalized exponential distribution hazardfunction, Ovarian Reserve,ovarian volume, Reproductive age***

I. INTRODUCTION

Ovarian Reserve :

Ovarian reserve is a term that is used to determine the capacity of the ovary to provide egg cells that are capable of fertilization resulting in a healthy and successful pregnancy. With advanced maternal age there can be declines, constituting a major factor in the inverse correlation between reproductive age and female fertility.

Age and Female fertility:

Female fertility is affected by age. Age is thus a major fertility factor for women. After puberty, female fertility increases and then decreases, with advanced maternal age causing an increased risk of female infertility. In human the ovary contains a fixed pool of primordial follicles, maximal at 5 months of gestational age, which declines with increasing age in a bi-exponential fashion, culminating in the menopause at an average age of 50-51 years. For any given age, the size of the follicle pool can be estimated based upon a mathematical model of decline. The rate of follicle decline represents an instantaneous rate of temporal change. Reproductive ageing in women is due to ovarian follicle depletion.

The rapid increase in the use of transvaginal sonography, the measurement of ovarian volume has become quick, accurate and cost-effective. Ovarian volume measurement has become a potentially useful tool in the screening, diagnosis and monitoring of the treatment of conditions such as polycystic ovarian syndrome and ovarian cancer, and in the prediction of super ovulation during IVF (Lass and Brinsden, 1999). The aim of this work is to describe a methodology for determining a woman's reproductive age and ovarian reserve by measurement of ovarian volume by transvaginal sonography.

II. METHODS AND RESULTS

a. Natural Decay of the Ovarian Follicle Pool and Solution of the Faddy-Gosden Equation

Graphical representation of ovarian follicle number, expressed logarithmically against age, suggests that ovarian follicle decline is bi-exponential. An increase in the rate of exponential decline appeared to occur at a follicle pool of ~ 25 000, corresponding to an average age of 37 years. However, biologically, an abrupt change when the primordial follicle population falls to ~ 25000 is unlikely, and more possibly the change represents an instantaneous rate of temporal change based on the remaining population pool, which is expressed mathematically as a differential equation. Faddy and Gosden (1996) provided a revised model in terms of the differential equation: dy/dx = -y [0.0595 + 316/ (11 780+ y)], where X denotes age and Y denotes primordial follicle population, with initial value y (0) = 701200. This equation expresses the rate of change in the population from birth.

Figure 1 a.

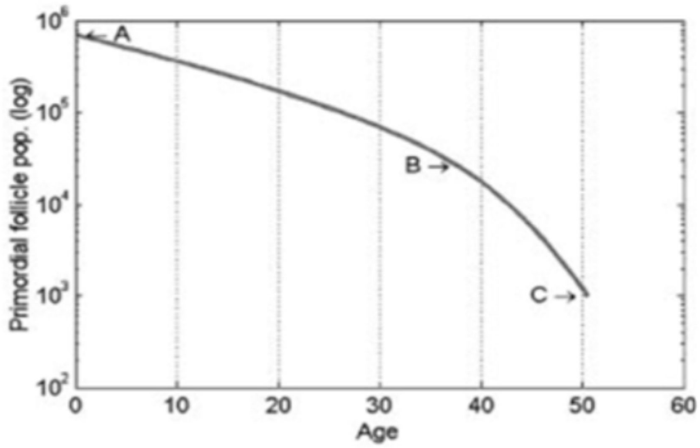

Figure 1. (a) The solution of the Faddy-Gosden differential equation for the primordial follicle population from birth to menopause. The primordial follicle population at birth is ~ 701 000 (A), and at menopause is ~ *1000* at 50.4 years (C) ,with an accelerated decline occurring at ~ 25 000 remaining primordial follicles (B).

b. Representation of Ovarian volume v/s Reproductive Age

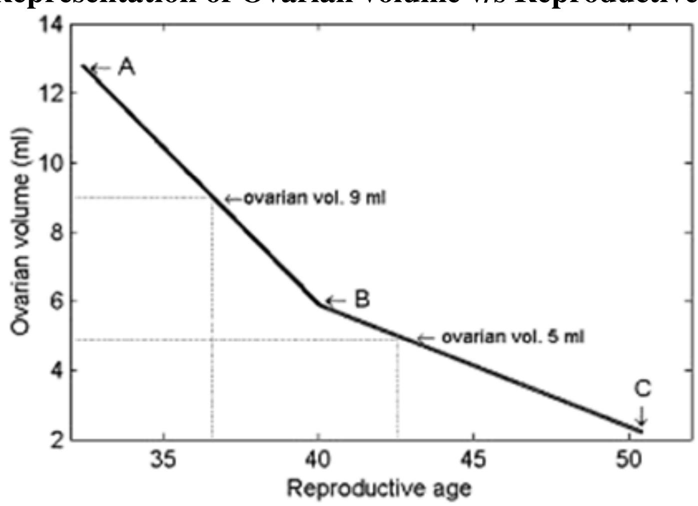

Figure 1.(b) For a given ovarian volume and chronological age, we estimate reproductive age from a plot through the three points

Point A : (32.4 years, 8.8 ml),

Point B (40 years,5.9 ml)

Point C (50.4 years, 2.2 ml)

In Figure 2, we describe the relationship between ovarian volumes (as a surrogate measure for the remaining primordial follicle pool) and reproductive age for women of chronological ages 25-45 years at 5 year intervals. The age at menopause for each age group is fixed at 50.4 years, corresponding to an ovarian volume of 2.2 ml being the mean ovarian volume for post-menopausal women. More specifically, if a woman of chronological age 25-51 years has her mean ovarian volume of measured by transvaginal sonography, then, taking ovarian volume data for her age , we can estimate her reproductive age using ovarian volume as a surrogate for her ovarian reserve. We can then predict her age of menopause. This prediction will only apply to women who have no evidence of ovarian disease who are not on hormonal contraception.

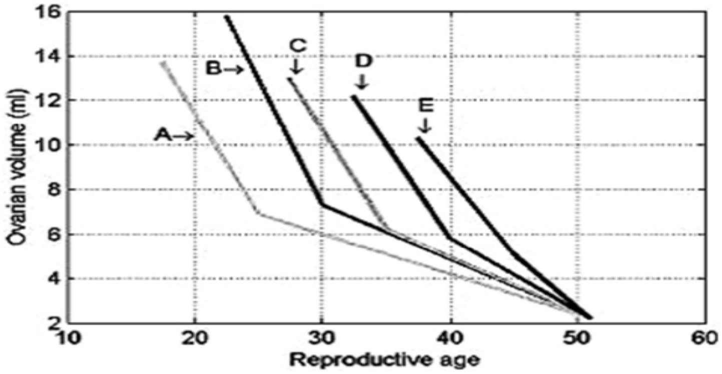

Figure 2.Reproductive age (years) versus volume (ml) for women of 25-45 years at 5 yearly intervals.
Line A, women of chronological age 25 years;
Line B, women of chronological age 30 years;
Line C, women of chronological age 35 years;
Line D, women of chronological age 40 years;
Line E, women of chronological age 45 years.
Number of years remaining before the menopause is 50-51 years minus reproductive age.

III. MATHEMATICAL MODEL FOR EXTENSION OF THE GENERALIZED EXPONENTIAL DISTRIBUTION

The two-parameter generalized exponential (GE) distribution has the following probability density function (PDF);

s $$f_{GE}(x;\alpha,\lambda) = \alpha\lambda e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\alpha-1}; \quad x>0. \qquad (1)$$

Here α> 0 and λ> 0 are the shape and scale parameters respectively. The two-parameter GE distribution has been used quite effectively for analysing lifetime data.

Although, GE distribution can be used quite effectively to analyse a data set which has monotone(increasing/decreasing)hazard function(HF),but unfortunatelyit cannotbe used ifthe HFisunimodal orbathtubshaped, similartotheWeibull orgammadistributions. The main aim of this paper is to extend the GE distribution to a three-parameter distribution, with an additional shape parameter. Many well-known distributions can be obtained as special cases of the proposed distribution. This new family of distribution functions is alwayspositively skewed,and the skewnessdecreasesasboth the shapeparametersincrease to infinity. Interestingly, the new three-parameter distribution has increasing, decreasing, uni-modal andbathtub shaped HFs. Therefore,it canbeusedquiteeffectivelyforanalysing different types of lifetime data.

Extended GE Family

The extended GE (EGE) family has the distribution function;

$$F(y;\alpha,\beta,\lambda) = \begin{cases} \left(1-(1-\beta\lambda y)^{\frac{1}{\beta}}\right)^{\alpha} & \text{if } \beta \neq 0 \\ \left(1-e^{-\lambda y}\right)^{\alpha} & \text{if } \beta = 0 \end{cases} \qquad (2)$$

for α> 0, λ> 0 and −∞ <β< ∞. The support of the EGE random variable Y in(2)is (0,∞)if β ≤ 0, and(0,1/(βλ))if β> 0. The following values of the parameters α and β are of particular interest;(i) β =0,EGE reduces to GE,(ii) β = 0, α = 1, EGE reduces to exponential, (iii) β = 1, α = 1, EGE reduces to uniform, (iv) α = 1, EGE reduces to generalized Pareto,(v)α = 1, β< 0, EGE reduces to Pareto. It may be mentioned that the generalized Pareto distribution has received considerable attention in the recent statistical literature because of its capability to model exceedances over

a threshold, see for example Johnson,Kotz and Balakrishnan[16] or Davison and Smith[7]. From now on the three-parameter EGE distribution with parameters α, β and λ will be denoted by EGE(α,β,λ).
The PDF of EGE(α,β,λ)becomes

$$f(y;\alpha,\beta,\lambda)=\begin{cases}\alpha\lambda\left(1-(1-\beta\lambda y)^{\frac{1}{\beta}}\right)^{\alpha-1}(1-\beta\lambda y)^{\frac{1}{\beta}-1} & \text{if} \quad \beta\neq 0\\ \alpha\lambda\left(1-e^{-\lambda y}\right)^{\alpha-1}e^{-\lambda y} & \text{if} \quad \beta=0,\end{cases} \tag{3}$$

when 0 <y< ∞ and 0 <y< 1/(βλ), for β ≤ 0 and β> 0, respectively. Moreover, its quantile function is

$$Q(u;\alpha,\beta,\lambda)=\begin{cases}\frac{1}{\beta\lambda}\left[1-(1-u^{1/\alpha})^{\beta}\right] & \text{if} \quad \beta\neq 0\\ -\frac{1}{\lambda}\ln\left(1-u^{1/\alpha}\right) & \text{if} \quad \beta=0.\end{cases} \tag{4}$$

Clearly, as β → 0, the quantile function of the EGE distribution tends to the quantile function of the GE distribution. For β> 0, the quantile function of the EGE distribution coincides with the quantile function of a transformed beta distribution. For β> 0, the support is on a finite interval. The shape of the PDF is (i) unimodalif α> 1 and0 <β< 1, (ii) an increasing function if α> 1 and β> 1,(iii)and ecreasing functionif0 <α < 1 and 0 <β < 1,(iv)abathtub shapedif0 <α < 1 and β> 1. (v)If α = 1, for 0 <β < 1itis a decreasing function and for β> 1, it is an increasing function.
Forβ< 0,ithasthe support on the who lerealline and the shape of the PDF is unimodal if α> 0. For β = 0, it is well known that the PDF is a decreasing function if 0 <α ≤ 1and it is unimodal if α> 1. The PDFs of the EGE for different ranges of α and β when λ =1 are plotted in Figure

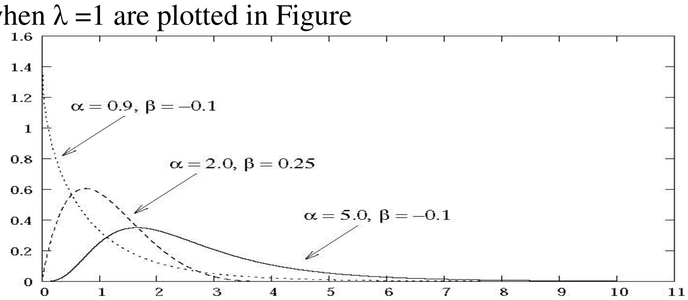

Figure 1: The PDFs of the extended GE for different values of α and β, when λ = 1

It is clear that the GE family has been embedded in a larger family, with an additional shape parameter β. Due to this additional shape parameter, more flexibility can be incorporated in the family, which will be useful for data analys is purposes. The EGE model can be seen as a proportional reversed hazard rate model (PRHRM). Therefore, several properties of the general PRHRM model can be easily translated for the EGE model. In the next section we discuss different structural properties of the EGE model.

A. Properties

The HF of EGE takes the form

$$h(y;\alpha,\beta,\lambda)=\begin{cases}\dfrac{\alpha\lambda\left(1-(1-\beta\lambda y)^{\frac{1}{\beta}}\right)^{\alpha-1}(1-\beta\lambda y)^{\frac{1}{\beta}-1}}{1-\left(1-(1-\beta\lambda y)^{\frac{1}{\beta}}\right)^{\alpha}} & \text{if} \quad \beta\neq 0\\ \dfrac{\alpha\lambda\left(1-e^{-\lambda y}\right)^{\alpha-1}e^{-\lambda y}}{1-\left(1-e^{-\lambda y}\right)^{\alpha}} & \text{if} \quad \beta=0.\end{cases} \tag{5}$$

It is observed that the HF can take all four different shapes namely (i) increasing, (ii) decreasing,(iii) uni modal and (iv) bathtub.

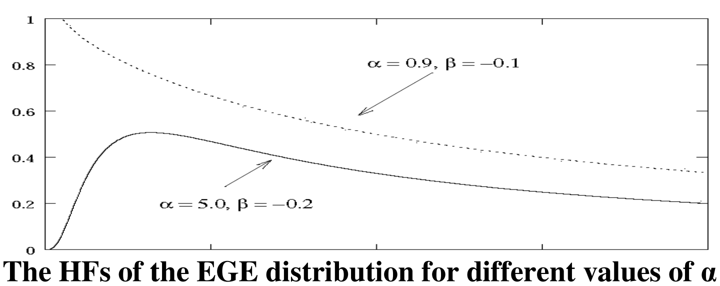

Figure 2: The HFs of the EGE distribution for different values of α and β< 0.

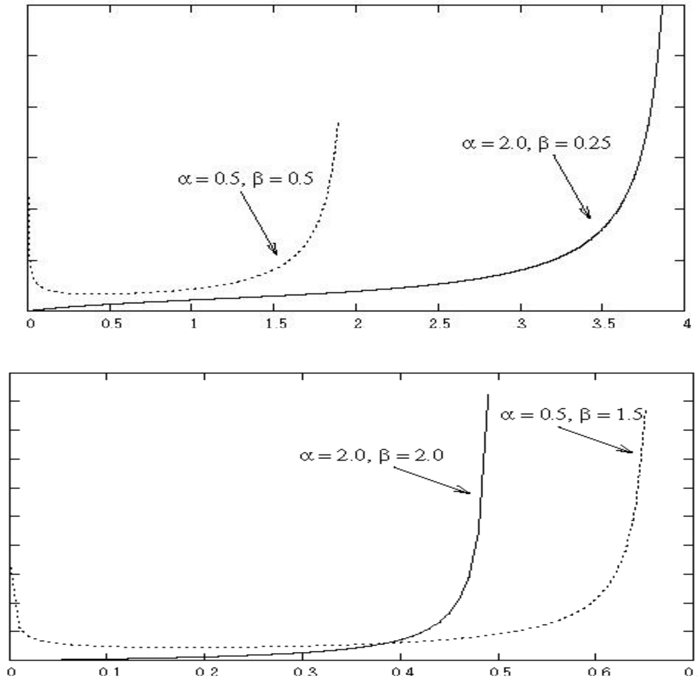

Figure 3: The HFs of the EGE distribution for different values of α and β> 0

The Figures 2 and 3 provide the HFs of the EGE distributions for different values of α and β, when λ = 1. We have the following results regarding the shapes of the HFs of the EGE distributions. The proofs are provided

Theorem 1: The HF of the EGE distribution is (a) uni modal if α> 1 and β< 0,(b) a decreasing function if α< 1 and β< 0.

Theorem 2: The HF of the EGE distributions(a) an increasing function if α> 1 and β> 1,(b) a bath tub shape dif α< 1 and β< 1.

IV. MATHEMATICAL RESULTS

For different values of shape and Scale parameters we have the following figure for the application part.

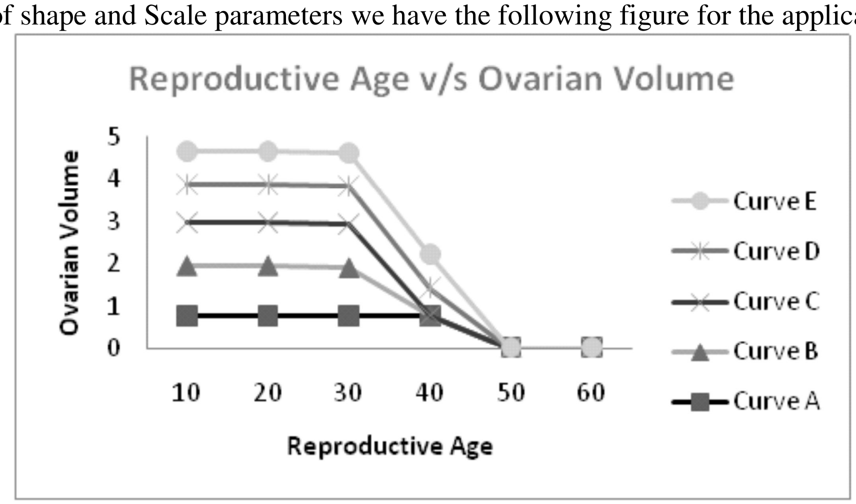

Fig.4 Reproductive ageing in women is due to ovarian follicle depletion.

V. CONCLUSION

We have shown that ovarian reserve and the reproductive age can be determined from the measurement of ovarian volume. There are two observations made:

1. That the observed variation in age at menopause is due to a wide difference in the primordial follicle population at birth and

2. That ovarian volume between the ages of 25 and 50 years is directly associated with the remaining primordial follicle population.

In this direction we have developed two parameter generalized exponential distribution to analyse a data set (Ovarian volume v/s Reproductive age) to obtain a monotonically decreasing Hazard Function (HF) which is always positively skewed and the skewness decreases as both the shape parameters increase to infinity. The mathematical figure determines the timing of the menopause with the declining curves of ovarian volume related to reproductive age showing statistically significant decrease in ovarian volume with each decade of life from 30 to 50 years.

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DATA MINING PROCESS AND TECHNIQUES

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ABSTRACT :
In the present communication data mining process is defined and illustrated with examples. Data mining techniques, namely, associate rule mining, classification and prediction, and clustering are discussed in details. A new concept of a measure of homogeneity of set of examples called entropy is introduced in order to define information gain. In the end application of techniques is discussed in brief.

1. INTRODUCTION

Data is collected in many different areas and the fast growth of the internet supports the overflow of data every day. The availability of information increases constantly, whereas the cost for storing this data decreases at the same time.

As an example the database of Wal-Mart included 11 terabyte of customer transactions already in 1998 and the quantity of world's data approximately doubles every year. Therefore, it is hard to extract subjectively important information from these huge amounts of data unless we have certain tool or method for analysis this huge abundant of data pool. These techniques and theories for understanding and analyzing data are subject of the Data Mining field.

Definition: Data mining is defined as exploring relationships and patterns in data to convert data into information and knowledge.

The data is majorly stored in databases, wherefore Data Mining is often synonymously used with the terms Knowledge Discovery in Databases (KDD) or Knowledge Mining. Knowledge discovery is commonly viewed as the non-trivial general process of discovering valid, novel, understandable, and ultimately useful knowledge about an application domain from observation data and background knowledge, in which the discovered knowledge is implicit or previously unknown.

Data mining techniques were developed in computer science rather than in statistics as these techniques differ in their parameters and methodologies of the exploration. The clustering technique is useful for data visualization and explorative data analysis. It builds clusters of similar data sets and tries to reveal previously unknown similarities between data sets. The human explorer has a major role in the clustering technique. Regression techniques try to predict the value of certain variables by exploring patterns in the data set. Based on this patterns the value of a variable is predicted by deriving it from the remaining ones using mathematical practices. This technique is useful for numerical data sets.

Data mining is different compared to other data analysis technique like structured queries that are used in many data bases. These techniques often act according to the same schema. A constructed hypothesis is proved or disproved using the data set. This verification-based approach has strict limitations like the creativity in the creation of the hypothesis or the construction of a good search query. Data Mining tries to overcome these weaknesses by analyzing multidimensional data sets and relations simultaneously. These advantages make Data Mining interesting both in public and private sectors, supported by the growth of computer networks and database connections to access huge amounts of data.

2. DATA MINING PROCESS

Data Mining is a process that starts with data and ends with previously unknown patterns and knowledge. This series of activities is divided into 5 steps as shown in Figure 1 regarding. The raw data is selected and analyzed during the steps to reveal patterns and create new knowledge.

Selection: The vast amount of raw data needs to be preselected for the following steps to decrease the overhead of data. The aims of the projects must be defined to select data based on features, observations and background knowledge and possibly highlight data deficiencies. Data understanding and background knowledge are crucial requirements for the selection phase. Uninteresting data sets can be sorted out if factors like quality, nature of the data and relation to the previously defined aim is specified. The process of data selection is mostly automated. The selected data is a subset of the overall data set that fulfills the request to support the sequencing steps.

Preprocessing: Data preprocessing is a heavy and time consuming task. The preselected data is verified to find inappropriate values and edited where required. During the verification step missing data may be discovered, as a result of not or wrongly measurements or instrument malfunctions. Missing values can be completed by human input, averaged values or fuzzy set values for example. The used techniques are closely related to the application and project aims. Data preprocessing creates a structured and complete data set.

Transformation: The transformation step is executed after the preprocessing to create a descriptive model of the data to enable a computer based processing. The model has to fit to the application but there are general requirements, every model should fit. It should be parsimonious on the one hand to decrease the computational costs and consider the important data values on the other hand. Dimension reduction is used to decrease the amount of data contemporaneously with keeping the content as similar as possible. A common technique for dimension reduction is Singular Value Decomposition (SVD). Some algorithms only work with normalized data, wherefore data transformation has to fulfill this request regarding to the chosen data mining technique.

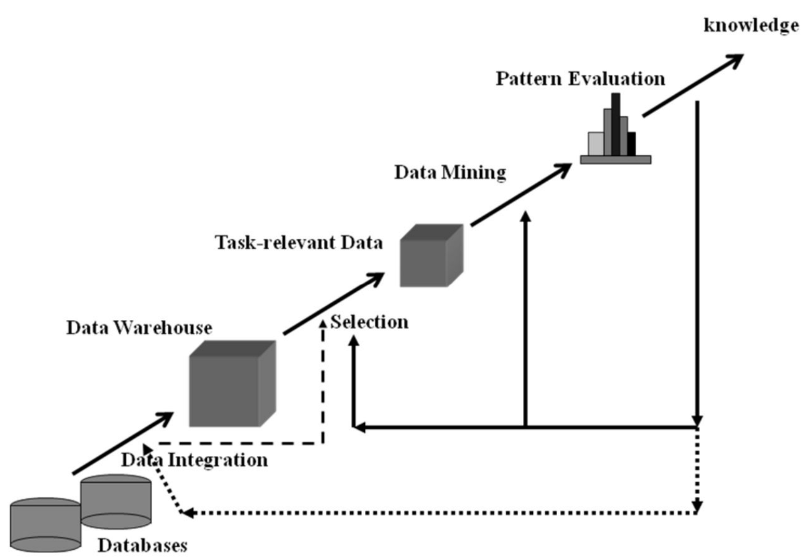

Figure-1

3. Data Mining: The Data Mining step in the overall process of Data Mining is for the recognition and Extraction of patterns from the transformed data set. A Data Mining technique (e.g. clustering) fitting best to the application requirements has to be chosen. This is of course one of the initial steps, since it affects the model building and first steps as well. This recognition can be based on algorithms or exploration of the visualized data set using Explorative Data Analysis. To guide the process of knowledge discovery, evaluation processes to determine the goodness of the results and develop enhancements are needed. This enables a dependable validation of the revealed patterns and supports the learning process.

Interpretation: The interpretation is the final step in the Data Mining process. The results and revealed patterns of the data mining process are interpreted to create new knowledge. Fixed schemas for data interpretation can hardly be formulated, so that human interaction is needed for reliable results. This phase is influenced by subjectivity and inter subjectivity can be created by increasing the amount of observers. The arrows in Figure 1 show that the results can influence every step of the overall process. If the accuracy of the results is not sufficient, it might be traced to a partly wrong defined model that then can be corrected in the Transformation phase, or a non fitting data selection algorithm chosen in the Selection step.

In order to find interesting pattern from pool of Data, data mining offers following technique

- Association rule mining/ frequent patterns mining
- Classification and prediction.
- Clustering techniques

4. FREQUENT PATTERN MINING (ASSOCIATION RULE MINING)

Frequent pattern mining has been a focused theme in data mining research for over a decade. Abundant literature has been dedicated to this research and tremendous progress has been made, ranging from efficient and scalable algorithms for frequent item set mining in transaction databases to numerous research frontiers, such as sequential pattern mining, structured pattern mining, correlation mining, associative classification, and frequent pattern-based clustering, as well as their broad applications. Here, we provide a brief overview of the current status of frequent pattern mining and discuss a few promising research directions. We believe that frequent pattern mining research has substantially broadened the scope of data analysis and will have deep impact on data mining methodologies and applications in the long run refer to [1].

The problem of association rule mining was introduced by R. Agrawal et al. (1993) which consists of finding the largest item set according to a user specified min-support (minimum support) and then finding the association rules according to a user specified min-confidence (minimum confidence).

Apriori algorithm (R. Agrawal et al., 1994) aims at finding the association rules in market basket data (MBD). The drawback of this algorithm is that it makes a complete pass over the MBD each time the support for item sets in the candidate item set C_k ,k=1,2,3,….n is to be counted for the purpose of generating a large item set L_k. [1]

The algorithm Apriori TID (Apriori Transaction Identifier) (R. Agrawal et al., 1994) is an enhancement over Apriori as it does not use the database again for counting support of itemsets after the first pass. But during early passes the sets obtained are comparable to the size of the database. Apriori performs better than Apriori TID during earlier passes but is slower than AprioriTID in later passes due to the decreasing size of the candidate itemsets.

A priori Hybrid (R.Agrawal et al., 1994) is a combination of A priori algorithm and uses A priori in the early passes and then switches to A priori TID during later passes. Additional cost is incurred while switching [1].

Psuedocode

Initialize: *k :=1,* C_1 *= all the 1-itemsets;*

read the database to count the support of C_1 *to determine* L_1*;*

L_1 := {frequent 1-itemsets};
k := 2; // k represents the pass number//
while *($L_{k-1} \neq \emptyset$)* ***do***
begin
C_k := ***gen_candidate_itemsets*** *with the given L_{k-1}*
prune(C_k)
for all *transactions $t \in T$* ***do***
increment the count of all transactions in C_k that are contained in t ;
L_k := All candidates in C_k minimum support ;
k := k+1 ;
end
gen_candidates_itemsets *with the given L_{k-1} as follows :*
$C_k = \emptyset$
for all *itemsets $l_1 \in L_{k-1}$* ***do***
for all *itemsets $l_2 \in L_{k-1}$* ***do***
if $l_1[1] \wedge l_1[2] = l_2[2] \wedge \ldots \wedge l_1[k-1] < l_2[k-1]$
  then $c = l_1[1], l_1[2] \ldots\ldots l_1[k-1], l_2[k-1]$
  $C_k = C_k \cup \{c\}$
prune(C_k)
for all $c \in C_k$
for all *(k-1) - subsets d of c* ***do***
  if $d \notin L_{k-1}$
  then $C_k = C_k \setminus \{c\}$

5. CLASSIFICATION TECHNIQUES

It refers to the data mining problem of attempting to predict the category of categorical data by building a model based on some predictor variables. Classification involves finding rules that partition the data into disjoint groups. The input for the classification is the training data set, whose class labels are already known. Classification analyzes the training data set and constructs a model based on the class label, and aims to assign a class label to the future unlabelled records [12]

The several classification discovery models are decision trees, neural networks, genetic algorithms and the statistical models like linear/geometric discriminates. **Decision Tree** generates a tree and a set of rules, representing the model of different classes from a given data set. The two disjoint subsets – a training set and a test set, are used. The training set is used for deriving the classifier while the test set is used to measure the accuracy of the classifier. The accuracy of the classifier is determined by the percentage of the test examples that are correctly classified.

Decision trees

Decision trees are powerful and popular tools for classification and prediction. The attractiveness of decision trees is due to the fact that, in contrast to neural networks, decision trees represent *rules*. Rules can readily be expressed so that humans can understand them or even directly used in a database access language like SQL so that records falling into a particular category may be retrieved.

A well known tree and frequently used over the years is C4.5 (or improved, but commercial version See5/C5.0).

Decision tree is a classifier in the form of a tree structure (Figure 2), where each node is either:

- a *leaf node* - indicates the value of the target attribute (class) of examples,
- a *decision node* - specifies some test to be carried out on a single attribute-value, with one branch and sub-tree for each possible outcome of the test.

A decision tree can be used to classify an example by starting at the root of the tree and moving through it until a leaf node, which provides the classification of the instance. Decision tree induction is a typical inductive approach to learn knowledge on classification. The key requirements to do mining with decision trees are:

- *Attribute-value description*: object or case must be expressible in terms of a fixed collection of properties or attributes. This means that we need to discrete continuous attributes, or this must have been provided in the algorithm.
- *Predefined classes (target attribute values)*: The categories to which examples are to be assigned must have been established beforehand (supervised data).
- *Discrete classes*: A case does or does not belong to a particular class, and there must be more cases than classes.
- *Sufficient data*: Usually hundreds or even thousands of training cases.

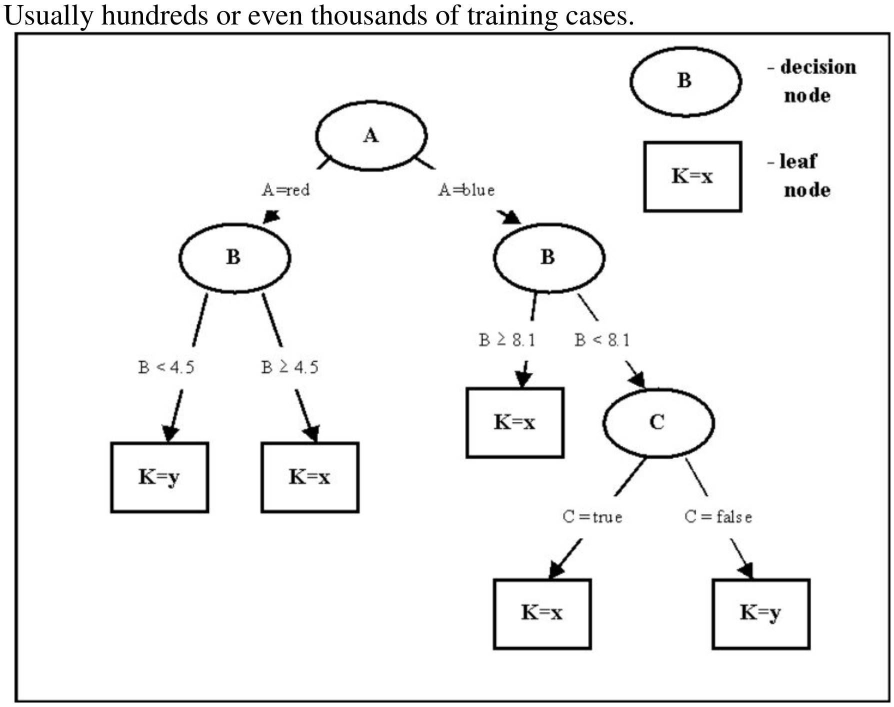

Figure 2: An example of a simple decision tree

CONSTRUCTING DECISION TREES

Most algorithms that have been developed for learning decision trees are variations on a core algorithm that employs a top-down, greedy search through the space of possible decision trees. Decision tree programs construct a decision tree *T* from a set of training cases.

J. Ross Quinlan originally developed ID3 at the University of Sydney. He first presented ID3 in 1975 in a book, *Machine Learning*, vol. 1, no. 1. ID3 is based on the Concept Learning System (CLS) algorithm

ID3 searches through the attributes of the training instances and extracts the attribute that best separates the given examples. If the attribute perfectly classifies the training sets then ID3 stops; otherwise it recursively operates on the m (where m = number of possible values of an attribute) partitioned subsets to get their "best" attribute. The algorithm uses a greedy search, that is, it picks the best attribute and never looks back to reconsider earlier choices. Note that ID3 may misclassify data.

The central focus of the decision tree growing algorithm is selecting which attribute to test at each node in the tree. For the selection of the attribute with the most inhomogeneous class distribution the algorithm uses the concept of entropy, which is explained next. [13]

1) ***The best classifier***

The estimation criterion in the decision tree algorithm is the selection of an attribute to test at each decision node in the tree. The goal is to select the attribute that is most useful for classifying examples. A good quantitative measure of the worth of an attribute is a statistical property called *information gain* that measures how well a given attribute separates the training examples according to their target classification. This measure is used to select among the candidate attributes at each step while growing the tree.

2) ***Entropy - a measure of homogeneity of the set of examples***

In order to define information gain precisely, we need to define a measure commonly used in information theory, called entropy that characterizes the purity of an arbitrary collection of examples. Given a set S, containing only positive and negative examples of some target concept (a 2 class problem), the entropy of set S relative to this simple, binary classification is defined as

$$\text{Entropy}(S) = - p_p \log_2 p_p - p_n \log_2 p_n, \qquad (5.1)$$

where p_p is the proportion of positive examples in S and p_n is the proportion of negative examples in S. In all calculations involving entropy we define 0log0 to be 0. [12]

Illustration: Suppose S is a collection of 25 examples including 15 positive and 10 negative examples [15+, 10-]. Then the entropy of S relative to this classification is

$$\text{Entropy}(S) = - (15/25) \log_2 (15/25) - (10/25) \log_2 (10/25) = 0.970 \qquad (5.2)$$

Notice that the entropy is 0 if all members of S belong to the same class. For example, if all members are positive (p_p= 1), then p_n is 0, and Entropy(S) = $-1 \log_2 (1) - 0 \log_2 0 = -1 \text{x } 0 - 0 \log_2 0 = 0$. Note the entropy is 1 (at its maximum!) when the collection contains an equal number of positive and negative examples. If the collection contains unequal numbers of positive and negative examples, the entropy is between 0 and 1. Figure 3 shows the form of the entropy function relative to a binary classification, as p_+ varies between 0 and 1.

Also Information gain is used to measure how good a tribute is for predicting the class of each of the training data.

The strengths of decision tree methods are:

- Decision trees are able to generate understandable rules.
- Decision trees perform classification without requiring much computation.
- Decision trees are able to handle both continuous and categorical variables
- Decision trees provide a clear indication of which fields are most important for prediction or classification.

  The weaknesses of decision tree methods are
- Decision trees are less appropriate for estimation tasks where the goal is to predict the value of a continuous attribute.
- Decision trees are prone to errors in classification problems with many classes and relatively small number of training examples.

- Decision tree can be computationally expensive to train. The process of growing a decision tree is computationally expensive. At each node, each candidate splitting field must be sorted before its best split can be found. In some algorithms, combinations of fields are used and a search must be made for optimal combining weights. Pruning algorithms can also be expensive since many candidate sub-trees must be formed and compared.
- Decision trees do not treat well non-rectangular regions. Most decision-tree algorithms only examine a single field at a time. This leads to rectangular classification boxes that may not correspond well with the actual distribution of records in the decision space.

Naïve Bayes Algorithm

The Naive Bayes algorithm is based on conditional probabilities. It uses Bayes' Theorem, a formula that calculates a probability by counting the frequency of values and combinations of values in the historical data. [13]

Bayes' Theorem finds the probability of an event occurring given the probability of another event that has already occurred. If B represents the dependent event and A represents the prior event, Bayes' theorem can be stated as follows.

Bayes' Theorem: Prob (B given A) = Prob (A and B)/Prob (A)

To calculate the probability of B given A, the algorithm counts the number of cases where A and B occur together and divides it by the number of cases where A occurs alone.

Naive Bayes makes the assumption that each predictor is conditionally independent of the others. For a given target value, the distribution of each predictor is independent of the other predictors. In practice, this assumption of independence, even when violated, does not degrade the model's predictive accuracy significantly, and makes the difference between a fast, computationally feasible algorithm and an intractable one.

Sometimes the distribution of a given predictor is clearly not representative of the larger population.

Advantages of Naive Bayes

The Naive Bayes algorithm affords fast, highly scalable model building and scoring. It scales linearly with the number of predictors and rows. The build process for Naive Bayes is parallelized. (Scoring can be parallelized irrespective of the algorithm.) Naive Bayes can be used for both binary and multiclass classification problems.

6. CLUSTERING TECHNIQUES

Data clustering is probably the most well known data and text mining technique. A set of observations is grouped so that a group is internally homogeneous and these groups among themselves are heterogeneous. Clustering techniques are divided into several groups that are described in this section together with algorithms.

Clustering techniques fall into a group of undirected data mining. The goal of undirected data mining is to discover structure in the data as a whole. There is no target variable to be predicted, thus no distinction is being made between independent and dependent variables. [8]

Clustering techniques are used for combining observed examples into clusters (groups) which satisfy two main criteria:

- Each group or cluster is homogeneous; examples that belong to the same group are similar to each other.
- Each group or cluster should be different from other clusters, that is, examples that belong to one cluster should be different from the examples of other clusters.

Depending on the clustering technique, clusters can be expressed in different ways:

- Identified clusters may be exclusive, so that any example belongs to only one cluster.
- They may be overlapping; an example may belong to several clusters.

- They may be probabilistic, whereby an example belongs to each cluster with a certain probability.
- Clusters might have hierarchical structure, having crude division of examples at highest level of hierarchy, which is then refined to sub-clusters at lower levels.

Partitioned Clustering

This clustering technique arranges n data objects into k partitions, where k<=n. The amount k of clusters has to be defined in advance. Data objects within a cluster are similar, whereas objects of different clusters are dissimilar. Similarity is mostly measured using a distance function, like e.g. Euclidean distance. [7]

K-Means Technique

K-Means is a centroid-based clustering technique that is implemented widely because of its simplicity Efficiency and scalability. The number of resulting clusters k has to be defined in advance. Initially k Cluster centers are located randomly. From now in every step, each data set is allocated to the centre with the minimal distance (e.g. Euclidean) and the centroids of each cluster are recalculated, corresponding to the new data points they consist of. This steps are repeated until no change in the allocation occurs any more (the criterion function converges). The k-Means technique minimizes the intra cluster variance, or squared error:

$$E = \sum_{I=1}^{K} * \sum_{P \in Ci} ||p - mi||^2 \qquad (6.1)$$

where there are k clusters C_i, i = 1, 2... k and m_i is the centroid of all the points' $p \in S_i$.

Simplicity as well as efficiency and scalability are main advantage of the k-Means technique. It works well, when the clusters are rather well separated from each other. Predefining the number k of clusters is a disadvantage. The random choice of initial cluster centers tends to different results with the same Data and so results in local optima. K-Means is rather sensitive to outliers and the resulting clusters used. [7]

Hierarchical Clustering

Hierarchical clustering techniques group the data into a tree of cluster, where every step is based on the preceding one. This happens either by agglomerative or divisive clustering. This method work in contrary directions and describe the abstract way of building clusters.

Agglomerative Clustering is a bottom-up strategy. In the initial state each data set is handled as a separate cluster. In every step of the process, the two closest clusters are merged together to a single one until only one cluster remains or a termination condition is satisfied. Divisive Clustering is a top-down strategy. In the initial state all data sets are included into one cluster. In every step the cluster is divided into smaller clusters. The process stops when every data set is in a single cluster, or a termination condition is satisfied.

Simplicity is an advantage of hierarchical clustering; it can easily be understood and implemented. A major disadvantage of this technique is as mentioned above the dependency on each previously taken step, which cannot be backtracked. No possibility of improvement of a taken clustering step keeps the technique quite inflexible. If the algorithm is implemented with simple similarity measures like single, complete or average link, the clusters tend to have a proper convex shape, which mostly does not reflect the data arrangement and supports outsiders. [8]

Density-Based Clustering

Density-based clustering defines clusters as regions with a high density of objects separated by regions with low density. The main process is to explore these two region types. A common technique is to partition the data set into non overlapping cells. The cells with a high density of data points are supposed to be cluster centers, whereas the boundaries between clusters fall into the regions with low density. [11]

Grid-Based Clustering

The Grid-based approach quantizes the data space into a finite number of grid cells. All cluster operations are then processed on this cell. This clustering technique is independent from the number of data sets and only dependents on the number of cells and has very fast processing time.

7. APPLICATION

Data mining techniques have been applied successfully in many areas from business to science to sports. Here we are presenting web mining application World Wide Web (WWW) is a vast repository of interlinked hypertext documents known as web pages. A hypertext document consists of both, the contents and the hyperlinks to related documents. Users access these hypertext documents via software known as web browser. It is used to view the web pages that may contain information in form of text, images, videos and other multimedia. The documents are navigated using hyperlinks, also known as Uniform Resource Locators (URLs). Though the concept of hypertext is much older but WWW was originated after Tim Berners-Lee, an English physicist who wrote a proposal using hypertext to link and access information in 1990. Since then websites were being created around the world using hypertext markup languages and connected through Internet. [14][15]

With the increase use of web or web related activity we have collected a large amount of data in the form of web log. So web mining is a technique for efficiently use of this pool of data with the help of following technique:

- Web Content Mining
- Web Structure mining
- Web usage mining

One of the main applications of data mining is the ranking given by different search engine to web site based upon topic of interest provided as keyword.

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ALTERNATIVE ALGORITHM FOR FUZZY QUADRATIC FRACTIONAL PROGRAMMING PROBLEM

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ABSTRACT :
In this paper we focus on a kind of quadratic programming with fuzzy numbers and variables. First by using a fuzzy ranking and arithmetic operation, we transform these problems to crisp model with nonlinear objective and linear constraints and then we solve this problem by alternative method and obtain an optimal solution. The technique is useful to apply on numerical problems, reduces the labor work and save valuable time.
Keywords: ***Trapezoidal fuzzy numbers, ranking method, optimal solution.***

1. INTRODUCTION

Several factors in the real world imply the increase in quadratic function optimization problem is particular type of nonlinear problem in which the target function may be a rate of two quadratic objective functions subject to a set of linear constraints. Such problems arise naturally in decision making when several rates have to be completed for the optimization at the same time. For example financial and corporate planning, hospital and health care plan, production planning. In the literature, quadratic fractional optimization has received much attention. It is the fundamental problem in optimization theory and practice.

There were several results on quadratic fractional optimization. Jain Sanjay [5] proposed Modified gauss elimination technique for separable nonlinear programming problem. Durga P. [2] presented a method in which a fuzzy multi objective. Nonlinear problem is reduced to crisp using ranking function and then the crisp problem is solved by fuzzy programming technique. Lachhwani K. [8] studied on multi-objective quadratic fuzzy optimization problem. This involves optimization on several objective function in the form of a numerator and denominator function on the other view. Abdulrahim B.K. [1] studied on quadratic fuzzy optimization problem via feasible direction development and modified simplex method. Sen S. [12] presented a piecewise linear approximation method to solve fuzzy separable quadratic optimization problem. Pramanik S.et al [10] used fuzzy goal programming technique to solve the multi objective quadratic optimization problem.

Gani and Kumar [3] proposed the principal pivoting method for solving fuzzy quadratic programming problem. Ghadle and Pawar[4] proposed an alternative method for Quadratic fractional programming problem in special case. Kheirfam [7] proposed a method for solving fully fuzzy Quadratic programming problems. Pandian and Jayalakshmi[9] solved linear fractional programming problems. Sharma and Singh [11] proposed a method to solve Quadratic fractional optimization through FGP approach. Jain Sanjay et al.[6] proposed Solution of fuzzy linear fractional programming problem.

2. PRELIMINARIES

2.1 Trapezoidal and triangular fuzzy numbers

If the membership function $f_{\tilde{A}}(x)$ is piecewise linear, then $\tilde{A}$ is said to be a trapezoidal fuzzy number. The membership function of a trapezoidal fuzzy number is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{w(x-a)}{b-a} & if\ x\epsilon[a,b] \\ w & if\ x \in [b,c] \\ \dfrac{w(d-x)}{d-c} & if\ x\epsilon[c,d] \\ 0 & otherwise \end{cases}$$

If w=1, then $\tilde{A} = (a,b,c,d,1)$ is a normalized trapezoidal fuzzy number and $\tilde{A}$ is a generalized or non-normal trapezoidal fuzzy number if $0 < w < 1$. the image of $\tilde{A} = (a,b,c,d;w)$ is given by$-\tilde{A} = (-d,-c,-b,-a;w)$.

In particular case if b=c, the trapezoidal fuzzy number reduces to a triangular fuzzy number given by$\tilde{A} = (a,b,d;w)$. The value of b corresponds with the mode or core and [a,d] with the support. If w=1, then $\tilde{A} = (a,b,d)$ is a normalized triangular fuzzy number $\tilde{A}$ is a generalized or non-normal triangular fuzzy number if $0 < w < 1$.

2.2 Properties of trapezoidal fuzzy number

Let $\tilde{A} = (a_1, b_1, c_1, d_1)$ and $\tilde{B} = (a_2, b_2, c_2, d_2)$ be two trapezoidal fuzzy numbers, then the fuzzy numbers addition and fuzzy numbers subtraction are defined as follows:

i. Fuzzy numbers addition of $\tilde{A}$ and $\tilde{B}$ is denoted by $\tilde{A} + \tilde{B}$ and is given by

$$\tilde{A} + \tilde{B} = (a_1, b_1, c_1, d_1) + (a_2, b_2, c_2, d_2) = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2)$$

ii. Fuzzy numbers subtraction of $\tilde{A}$ and $\tilde{B}$ is denoted by $\tilde{A} - \tilde{B}$ and is given by

$$\tilde{A} - \tilde{B} = (a_1, b_1, c_1, d_1) - (a_2, b_2, c_2, d_2) = (a_1 - d_2, b_1 - c_2, c_1 - b_2, d_1 - a_2)$$

2.3 New approach for ranking of trapezoidal fuzzy numbers

2.3.1 Method of magnitude

If $\tilde{a} = (a_1, a_2, a_3, a_4)$ is a trapezoidal fuzzy number, then the defuzzified value or the crisp number of $\tilde{a}$ is given by $\tilde{a} = \frac{a_1+2a_2+2a_3+a_4}{6}$. We need the following definitions of ordering on the set of the fuzzy numbers based on the magnitude of a fuzzy number.

The magnitude of the trapezoidal fuzzy number, $\tilde{u} = (x_0 - \sigma, x_0, y_0, y_0 - \beta)$ with parametric form $\tilde{u} = \left(\underline{u}(r), \overline{u}(r)\right)$ where $\underline{u} = x_0 - \sigma + \sigma r\ and\ \overline{u} = y_0 + \beta - \beta r$ is defined as

Mag (u) $=\frac{1}{2}\left(\int_0^1 \left(\underline{u}(r) + \overline{u}(r) + x_0 + y_0\right) f(r) dr\right)$,

Where the function f(r) is a non-negative and increasing function on [0,1], with f(0)=0,f(1)=1 and $\int_0^1 f(r)dr = \frac{1}{2}$.

Obviously function f(r) can be considered as a weighting function. In actual applications, function f(r) can be chosen according to the actual situation. The magnitude of a trapezoidal fuzzy number u which is defined by (1) synthetically reflects the information on every membership degree and meaning of this magnitude is visual and natural. The resulting scalar value is used to rank the fuzzy numbers. In the other words Mag ($\tilde{u}$) is used to rank fuzzy numbers. The larger Mag ($\tilde{u}$) means the larger fuzzy number.

The magnitude of the trapezoidal fuzzy number $\tilde{u} = (a,b,c,d)$ is given by Mag$(\tilde{u}) = \frac{a+5b+5c+d}{12}$

or Mag $(\tilde{u}) = \frac{5}{12}(b+c) + \frac{1}{12}(a+d)$.

Let $\tilde{u}$ and $\tilde{v}$ be two trapezoidal fuzzy numbers. The ranking of $\tilde{u}$ and $\tilde{v}$ by the Mag$(\tilde{u})$, the set of trapezoidal fuzzy numbers is defined as follows:

i. $Mag(\tilde{u}) > Mag(\tilde{v})\ if\ and\ only\ if\ \tilde{u} > \tilde{v}$;

ii. $Mag(\tilde{u}) < Mag(\tilde{v})$ *if and only if* $\tilde{u} < \tilde{v}$ and

iii. $Mag(\tilde{u}) = Mag(\tilde{v})$ *if and only if* $\tilde{u} = \tilde{v}$;

The ordering $\geq$ and $\leq$ between any two trapezoidal fuzzy numbers $\tilde{u}$ and $\tilde{v}$ are defined as follows:

i. $\tilde{u} \geq \tilde{v}$; *if and only if* $\tilde{u} > \tilde{v}$ *or* $\tilde{u} = \tilde{v}$ *and*

ii. $\tilde{u} \leq \tilde{v}$; *if and only if* $\tilde{u} < \tilde{v}$ *or* $\tilde{u} = \tilde{v}$.

The magnitude approach for ranking fuzzy numbers has some mathematical properties. It does not imply much computational effort and does not require a priori knowledge of the set of all alternatives.

3. PROBLEM FORMULATION

Quadratic programming is one of the most important optimization techniques in operation research. In real world applications, quadratic programming models usually are formulated to find some future course of action. In the real life problems there may exists uncertainty about the parameters. In such a situation the parameters of quadratic programming problems may be represented as fuzzy numbers. We consider a special case of quadratic programming problem in which the quadratic fractional objective function can be converted into linear fractional objective function with fuzzy variables.

Mathematically, consider the special case of quadratic fuzzy fractional programming problem (QFFPP):

Maximize $Z=\dfrac{(C_B x_{\mathbf{B}} \oplus \alpha)(C'_B x_B \oplus \beta)}{(D_B x_B \oplus \gamma)(D'_B x_B \oplus \delta)}$

Subject to constraints: $Ax \leq b, x \geq 0$

Where all $C_B, C'_B, D_B, D'_B, \alpha, \beta, \gamma, \delta$ are trapezoidal fuzzy numbers.

4. ALTERNATIVE ALGORITHM FOR SPECIAL CASE OF FUZZY QUADRATIC FRACTIONAL PROGRAMMING PROBLEM

Find optimal solution of special case of QFFPP by an alternative method, algorithm is given as follows:

Step 1. Convert all fuzzy numbers into crisp value by using proposed ranking method.

Step 2. Check objective function of QFFPP is of maximization. If it is to be minimization type than convert it into maximization.

Step 3. Convert quadratic fractional objective function to the product of linear fractional objective functions.

Step 4. Check whether all b_i (RHS) are non- negative. If any b_i is negative then convert it to positive.

Step 5. Express the given QFFPP in standard form then obtain an initial basic feasible solution.

Step 6. Find net evaluation Δ_j for each variable x_j by the formula:

$$\Delta_j = \sum_{i=1}^{4} z^i \Delta_{ij}$$

$$z^1 = (C_B x_B + \alpha),\ z^2 = (C'_B x_B + \beta),\ z^3 = (D_B x_B + \gamma),\ z^4 = (D'_B x_B + \delta)$$
$$\Delta_{1j} = (C_B x_B - C_j),\ \Delta_{2j} = (C'_B x_B + C'_j),\ \Delta_{3j} = (D_B x_B + D_j),\ \Delta_{4j} = (D'_B x_B + D'_j)$$

Step 7. Use usual simplex method for this table and go to next step.

Step 8. Check solution for optimality if all $\Delta_{ij} \geq 0$, then current solution is an optimal solution, otherwise go to step 6 and repeat the same procedure.

Thus optimum solution of special type of QFFPP is obtained.

5. NUMERICAL EXAMPLE

$$Max\,Z = \frac{\{(0,1,1,2)x_1 + (2,2,2,2)\}\{(0,1,1,2)x_1 + (1,1,1,1)x_2 + (1,1,1,1)\}}{\{(-3,1,1,5)x_1 + (-2,2,2,6)x_2 + (3,3,3,3)\}\{(0,1,1,2) + (3,3,3,3)\}}$$

Subject to

$$(-2,2,6,10)x_1 + (-2,2,2,6)x_2 \le (2,6,10,14)$$

$$(0,1,1,2)x_1 + (2,2,2,2)x_2 \le (0,4,8,12)$$

$$x_1, x_2 \ge 0$$

By using the method for defuzzyfing the trapezoidal fuzzy numbers

$$mag(u) = \frac{5}{12}(b+c) + \frac{1}{12}(a+d)$$

$$R(0,1,1,2) = \frac{5}{12}(1+1)\frac{1}{12}(0+2) = 1$$

$$R(2,2,2,2) = 2$$

$$R(1,1,1,1) = 1$$

$$R(-3,1,1,5) = 1$$

$$R(-2,2,2,6) = 2$$

$$R(3,3,3,3) = 3$$

$$R(2,6,10,14) = 8$$

$$R(0,4,8,12) = 6$$

$$Max\,Z = \frac{(x_1+2)(x_1+x_2+1)}{(x_1+2x_2+3)(x_2+3)}$$

Subject to

$$4x_1 + 2x_2 \le 8$$

$$x_1 + 2x_2 \le 6$$

$$x_1, x_2 \ge 0$$

Now solving the above problem by an alternative method the detailed process of the solution is as follows.

QFFPP is in standard form

$$Max\,Z = \frac{(x_1+2)(x_1+x_2+1)}{(x_1+2x_2+3)(x_2+3)}$$

Subject to

$$4x_1 + 2x_2 + s_1 = 8$$

$$x_1 + 2x_2 + s_2 = 6$$

$$x_1, x_2, s_1, s_2 \ge 0$$

Where s_1, s_2 are slack variables.

By solving above problem, we obtain the following final simplex table.

						1	0	0	0
						1	1	0	0
						1	2	0	0
						0	1	0	0
C_B	C'_B	D_B	D'_B	X_B	b	x_1	x_2	s_1	s_2
1	1	1	0	x_1	$2/3$	1	0	$1/3$	$-1/3$
0	1	2	1	x_2	$8/3$	0	1	$-1/6$	$2/3$
					$z^1 = 8/3$	0	0	$1/3$	$-1/3$
					$z^2 = 13/3$	0	0	$1/6$	$1/3$
					$z^3 = 9$	0	0	0	1
					$z^4 = 17/3$	0	0	$-1/6$	$2/3$
					Δ_j	0	0	$2/3$	$40/3$

$\because$ All $\Delta_j \geq 0$ current solution is an optimal solution.

$\therefore$ Optimum solution is $x_1 = \dfrac{2}{3}, x_2 = \dfrac{8}{3} \quad Z = 0.226$.

CONCLUSION:

An alternative method to the solution of special case of fuzzy quadratic fractional programming problem has suggested. A number of algorithm have been developed to solve such type of FQFPP, each applicable to specific type our approach is general purpose to solve FQFPP and reduce number of iteration by selecting pivot element also it gives more efficiency.

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A STOCHASTIC MODEL FOR EXPECTED TIME TO RECRUITMENT IN A SINGLE GRADE MANPOWER SYSTEM UNDER CORRELATED WASTAGES AND CORRELATED INTER-DECISION TIMES

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ABSTRACT :
In this paper, a single graded marketing organization subjected to random exit of personal due to policy decisions taken by the organization is considered. The model is constructed by assuming the loss of man-hours and inter-decision times are exchangeable and constantly correlated exponential random variables. Mean and variance of time to recruitment are obtained using uni-variate policy based on shock model approach. The analytical results are numerically illustrated by assuming different distributions for the thresholds.
Keywords: ***Manpower planning, Shock model, Correlated inter-decision times, Univariate recruitment policy, Mean and variance of time to recruitment.***
2000 MSC Subject Classification: 90B70, 90B40, 91D35.

INTRODUCTION

An exit of personnel is a common phenomenon in any marketing organization. In [1] and [2] several stochastic models for a manpower system with grades are discussed using Markovian and renewal theoretic approach. Since several authors [3]-[7] contributed to the development of problem of time to recruitment in a single graded marketing organization involving single threshold under different conditions. Muthaiyan et. al. [6] have introduced and studied the system characteristics, that is, mean and variance of time to recruitment when the inter-decision times form an order statistics. In [8], Expected time to recruitment in single grade system using univariate policy is obtained assuming the amount of wastage at each decision epoch forms a sequence of exchangeable and constantly correlated exponential random variables and the inter-decision times form a sequence of order statistics. In the present paper the results in [8] are obtained when the inter-decision times are exchangeable and constantly correlated exponential random variables.

MODEL DESCRIPTION AND ANALYSIS :

Consider an organization having single grade in which decisions are taken at random epochs in $[0,\infty)$ and at every decision making epoch a random number of persons exit from the organization. There is an associated loss of man-hour to the organization, if a person exit and it is linear and cumulative. Let X_i be the loss of man-hours due to the i^{th} decision epoch $i = 1,2,3,\dots.k$ with cumulative distribution function $G(.)$ and probability density function $g(.)$.Let $X_i's$ be identically distributed, exchangeable and constantly correlated exponential random variables. Let ρ be the correlation (constant) between any X_i and X_j , $i \neq j$. Let the inter-decision times U_i are identically distributed, exchangeable and constantly correlated exponential random variables with cumulative distribution function $F(.)$ and probability density function $f(.)$. Let Y be a random variable denoting the threshold for the loss of man-hours with cumulative distribution function $H(.)$ and probability density function $h(.)$. Let T

be the time to recruitment in the organization with cumulative distribution function $L(.)$, probability density function $l(.)$, mean E(T) and variance V(T). Let $F_k(.)$ be the k fold convolution of $F(.)$. Let $l^*(.)$ and $f^*(.)$ be the Laplace transform of $l(.)$ and $f(.)$. The univariate recruitment policy employed in this paper is *'if the total loss of man-hours exceeds the mandatory threshold Y, the recruitment is necessary'.*

MAIN RESULTS:

The probability that the threshold level is not reached till t is given by

$$P[T > t] = \sum_{k=0}^{\infty}[F_k(t) - F_{k+1}(t)]\, P\left[\sum_{i=1}^{k} X_i < Y\right] \quad (1)$$

By using the law of total probability and on simplification we get

$$P\left[\sum_{i=1}^{k} X_i < Y\right] = \int_0^{\infty} P(Y > x)\, G_k(y)dy \ = \int_0^{\infty} G_k(y)h(y)dy \quad (2)$$

Case (i):

Suppose that the threshold Y follows exponential distribution with parameter θ.

$$P[T > t] = \sum_{k=0}^{\infty}[F_k(t) - F_{k+1}(t)] \int_0^{\infty} G_k(y)\, \theta\, e^{-\theta y} dy \quad (3)$$

If $X_i's$ are identically distributed, exchangeable and constantly correlated exponential random variables with mean b then the cumulative distribution function $G(.)$ of the partial sum is given in [3] as

$$G_k(y) = (1-\rho)\sum_{i=0}^{\infty} \frac{(k\rho)^i \Phi(k+i, y/b)}{(1-\rho+k\rho)^{i+1}\, (k+i-1)!} \quad (4)$$

where ρ is the correlation (constant) between any X_i and X_j , $i \neq j$.

From (3) and (4) it is found that

$$P[T > t] = \sum_{k=0}^{\infty}[F_k(t) - F_{k+1}(t)]\, (1-\rho) \sum_{k=0}^{\infty} A_k \quad (5)$$

where $A_k = \frac{1}{(1+b\theta)^{k-1}\, [(1-\rho)(1+b\theta)+kb\rho\theta]}$ and $b = \alpha(1-\rho)$

Here U_i's form a sequence of exchangeable and constantly correlated random variable having exponential distribution with p.d.f. $f(.)$ and c.d.f. $F(.)$

$$f(U) = \mu e^{-\mu u}, \mu > 0, 0 < u < \infty \quad (6)$$

The distribution function of $Z_k = \sum_{i=1}^{k} U_i$ is given in [3] and σ is the correlation (constant) coefficient between U_i and U_j , $i \neq j$

$$F_k(U_i) = P[Z_k \leq u] = (1-\sigma)\sum_{i=0}^{\infty} \frac{(\sigma k)^i \eta(k+i, u/m)}{(1-\sigma+k\sigma)^{i+1}\, (k+i-1)!} \quad \text{where } m = \beta(1-\sigma) \quad (7)$$

The Laplace Transform of the density function of Z_k is given by

$$f_k^*(s) = \frac{1-\sigma}{(1+ms)^{k-1}\, [(1-\sigma)(1+ms)+k\sigma ms]} \quad (8)$$

The distribution function of T is

$$L(t) = 1 - P[T > t] = 1 - (1-\rho)\sum_{k=0}^{\infty}[F_k(t) - F_{k+1}(t)]\, A_k \quad (9)$$

The density function of T is given by

$$l(t) = -(1-\rho)\sum_{k=0}^{\infty}[f_k(t) - f_{k+1}(t)]\, A_k \quad (10)$$

The Laplace transform of density function of T is given by

$$l^*(s) = -(1-\rho)\sum_{k=0}^{\infty}[f_k^*(s) - f_{k+1}^*(s)]\, A_k \quad (11)$$

Therefore from (8),

$$\frac{d}{ds}[f_k^*(s)]|_{s=0} = (1-\sigma)\left\{\frac{(-1)[(1-\sigma)m+k\sigma m]}{(1-\sigma)^2} + \frac{(1-k)m}{1-\sigma}\right\} = -k\beta \quad (12)$$

and $\frac{d}{ds}[f_{k+1}^*(s)]|_{s=0} = -(k+1)\beta$ (13)

The mean time to recruitment is given by

$$E(T) = -\frac{d}{ds} l^*(s)|_{s=0} \quad (14)$$

Using (12) and (13) in (14) we get,

$E(T) = \beta(1-\rho)\sum_{k=0}^{\infty} A_k$ (15)

Now, $\frac{d^2}{ds^2}[f_k^*(s)]_{s=0} = \beta^2[k(1-\sigma^2)+k^2(1+\sigma^2)]$

(16)

and $\frac{d^2}{ds^2}[f_{k+1}^*(s)]_{s=0} \quad = \beta^2[k[3+\sigma^2]+k^2[1+\sigma^2]+2]$ (17)

$E(T^2) = \frac{d^2}{ds^2} l^*(s)|_{s=0}$ (18)

Using (16) and (17) in (18) we get,

$E(T^2) = 2\beta^2\sum_{k=0}^{\infty}[k(1+\sigma^2)+1](1-\rho)A_k$ (19)

Case (ii):

Suppose that the mandatory threshold Y follows an extended exponential distribution with shape parameter 2.

The survival function of T is given by

$P(T>t) = 1-\frac{(1-\rho)}{\theta}\sum_{k=0}^{\infty}[F_k(t)-F_{k+1}(t)]\,[2A_k-\frac{1}{2}B_k]$ (20)

where $A_k = \frac{1}{(1-b\theta)^{k-1}\,[(1-\rho)(1+b\theta)+kb\rho\theta]}$ and $B_k = \frac{1}{(1+b2\theta)^{k-1}\,[(1-\rho+k\rho)(1+b2\theta)-k\rho]}$

Proceeding as in case (i), the following results are obtained.

$E(T) \;= \frac{\beta(1-\rho)}{\theta}\sum_{k=0}^{\infty}[2A_k-\frac{1}{2}B_k]$ (21)

$E(T^2) = \frac{2(1-\rho)\beta^2}{\theta}\sum_{k=0}^{\infty}[k(1+\sigma^2)+1][2A_k-\frac{1}{2}B_k]$ (22)

Case (iii):

Suppose that the mandatory threshold has Setting Clock Back to Zero (SCBZ) property.

In this case, the survival function of T is given by

$P(T>t) = 1-(1-\rho)\sum_{k=0}^{\infty}[F_k(t)-F_{k+1}(t)]\,[pC_k+qD_k]$ (23)

Where $C_k = \frac{1}{(1+b(\theta_1+\lambda))^{k-1}\,[(1-\rho+k\rho)(1+b(\theta_1+\lambda))-k\rho]}$ and $D_k = \frac{1}{(1+b\theta_2)^{k-1}\,[(1-\rho+k\rho)(1+b\theta_2)-k\rho]}$

$p = \frac{\theta_1-\theta_2}{\lambda+\theta_1-\theta_2}\ and\ q = \frac{\lambda}{\lambda+\theta_1-\theta_2}\ with\ p+q=1$

Proceeding as in case (i), we obtain,

$E(T) = \frac{\beta(1-\rho)}{\lambda}\sum_{k=0}^{\infty}[pC_k+qD_k]$ and (24)

$E(T^2) = \frac{2(1-\rho)\beta^2}{\lambda^2}\sum_{k=0}^{\infty}[k(1+\sigma^2)+1][pC_k+qD_k]$ (25)

In all the three cases the variance of the time to recruitment can be obtained by using $(T) = E(T^2)-[E(T)]^2$.

NUMERICAL ILLUSTRATION

The influence of model parameters on the performance measures namely mean and variance of the time to recruitment is studied numerically.

Table – 1 (Effect of ρ on the performance measures $E(T)$ and $V(T)$)

The values $\theta = 0.4,\ b=2,\ \beta=1,\ k=2,\ \lambda=1, \theta_1=0.4,\ \theta_2=0.2$ and $\sigma=0.2$ are fixed and varying ρ.

ρ	Case (i)		Case (ii)		Case (iii)	
	$E(T)$	$V(T)$	$E(T)$	$V(T)$	$E(T)$	$V(T)$
-0.6	0.4630	2.6375	1.9712	8.2570	0.5794	3.2332
-0.3	0.3883	2.2411	1.6832	7.5353	0.5071	2.8667
0	0.3086	1.8060	1.3583	6.5221	0.4366	2.4989
0.3	0.2235	1.3268	0.9965	5.1454	0.3486	2.0259
0.6	0.1323	0.7973	0.5964	3.3181	0.2325	1.3781

Table – 2 (Effect of k on the performance measures $E(T)$ and $V(T)$)

The values $\theta = 0.4,\ b = 2,\ \beta = 1,\ \rho = 0.9,\ \lambda = 1, \theta_1 = 0.4,\ \theta_2 = 0.2$ and $\sigma = 0.2$ are fixed and varying k.

k	Case (i)		Case (ii)		Case (iii)	
	$E(T)$	$V(T)$	$E(T)$	$V(T)$	$E(T)$	$V(T)$
1	0.1111	0.4410	0.4820	1.7343	0.1724	1.0323
2	0.0343	0.2100	0.1562	0.9378	0.0700	0.4263
3	0.0132	0.1085	0.0619	0.5063	0.0350	0.2144
4	0.0056	0.0578	0.0268	0.2759	0.0192	0.1179
5	0.0025	0.0312	0.0122	0.1512	0.0090	0.0553

Table – 3 (Effect of σ on the performance measures $E(T)$ and $V(T)$)

The values $\theta = 0.4,\ b = 2,\ \beta = 1,\ k = 2, \rho = 0.2,\ \lambda = 1, \theta_1 = 0.4,$ and $\theta_2 = 0.2$ are fixed and varying σ.

σ	Case (i)		Case (ii)		Case (iii)	
	$E(T)$	$V(T)$	$E(T)$	$V(T)$	$E(T)$	$V(T)$
-0.6	0.2841	1.2403	0.7007	2.7675	0.2378	1.0490
-0.3	0.3497	1.5883	0.8625	3.4755	0.2926	1.3459
0	0.4545	2.5207	1.1212	5.4701	0.3804	2.1377
0.3	0.6494	5.4781	1.6017	11.9873	0.5434	4.6421
0.6	1.1364	19.8450	2.8030	44.2790	0.9510	16.7842

FINDINGS

1. From the table 1, the mean and variance of time to recruitment decreases for all cases when the correlation coefficient ρ of wastages increases.
2. From the table 2, if $'k'$ the number of decision epochs in (0,t] increases, the mean and variance of time to recruitment decreases for all cases.
3. From the table 3, the mean and variance of time to recruitment increases for all cases when the correlation coefficient σ of inter-decision times increases.

CONCLUSION

From the numerical illustration we conclude that the mean and variance of the time to recruitment fully depends on the values of ρ, k and σ.

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A STUDY OF NON AUTONOMOUS EQUATIONS IN THE THEORY OF GYROSCOPES

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ABSTRACT :

This paper compares the efficiency of two methods via. method of splitting and method of unitary transformation in the analysis of real model system in theory of gyroscopes.

Keywords: ***gyroscope, splitting method, method of unitary transformations, stability.***

INTRODUCTION

In study small fluctuations of a micromechanical gyroscope [1] by means of non autonomous option of a method of splitting [2] in the presence of the vibrating basis are studied have an appearance:

$$\begin{aligned} &\ddot{\alpha} + 2\varepsilon\sigma\dot{\alpha} - \varepsilon b_1(t)\dot{\beta} + \omega^2\alpha = 0 \\ &\ddot{\beta} + 2\varepsilon\sigma\dot{\beta} - \varepsilon b_2(t)\dot{\alpha} + \omega^2\beta = 0 \\ &b_j(t) = b_j(1 + b_0 \sin \nu t), \end{aligned} \tag{1}$$

where α, β are the generalized coordinates describing the position of the sensor relative to the base; ω − characteristic frequency of the natural oscillations of the sensing element; σ − defined parameters of gyroscope; ν − oscillation frequency to the base; b_0 - amplitude oscillations of the base.

Equation (1) in vector form can be expressed as

$$\dot{x} = \left(A_0 + \varepsilon A_1(t)\right)x = A(t,\varepsilon)x, \tag{2}$$

where $x = (\alpha, \dot{\alpha}, \beta, \dot{\beta})^T$; $A_0 = \begin{pmatrix} A_{00} & 0 \\ 0 & A_{00} \end{pmatrix}$; $A_{00} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}$;

$$A_1 = \begin{pmatrix} A_{11} & A_{12}(t) \\ A_{21}(t) & A_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2\sigma & 0 & b_1(t) \\ 0 & 0 & 0 & 0 \\ 0 & -b_1(t) & 0 & -2\sigma \end{pmatrix}$$

and structure of a matrix of $A(t,\varepsilon)$ will allow to apply new asymptotic option of a method of splitting to non autonomous regularly indignant systems [2].

Theorem 1. *System* $\dot{x} = (A_0 + \varepsilon A_1(t))x,\ (x \in R^n \quad A(t,\varepsilon) = A_0 + \sum_{k=1}^{\infty} A_k(t)\varepsilon^k,\ |\varepsilon| \le \varepsilon_0 \ll 1$ with a T-periodic matrix of $A(t,\varepsilon)$ in a case when the range $\{\lambda_{0j}\}_1^n$ satisfies matrix of A_0 to inequalities:

$$\sigma_{jk} \equiv \lambda_{0j} - \lambda_{0k} \neq 2\pi q T^{-1}, \quad (j,k = \overline{1,n};\ j \neq k;\ q = 0, \pm 1, \pm 2, \ldots.),$$

by means of non- degenerate at | ε | ≪ 1) from T-periodic replacement yield

$$x = S_0(E + \textstyle\sum_1^N H_k(t)\varepsilon^k)z \tag{3}$$

where $S^{-1}A_0 = \Lambda_0 = diag\{\lambda_{01}, \ldots. \lambda_{0k}\}$, *a* $H_k(t)$ *T*-periodic matrix is transformed to simpler form as mentioned below.

$$\dot{z} = Q(t,\varepsilon)z, \quad \left(Q(t,\varepsilon) = \Lambda(\varepsilon) + \underline{\underline{O}}(\varepsilon^{N+1}), = \textstyle\sum_{k=0}^{N} \Lambda_k\, \varepsilon^N\right), \tag{4}$$

Here diagonal matrixes Λ_k and T-periodic $H_k(t)$ of a matrix unambiguously decide by k on the help of iterative algorithm.

Theorem 2. If in conditions Theorem 1, a vector $\{\lambda_j(\varepsilon)\}_1^n$ $j = \overline{1,n}$ auxiliary diagonal matrixes $\Lambda(\varepsilon)$ meets conditions

$Re\lambda_j(\varepsilon) \le -\sigma\varepsilon^q$ $(q = \overline{0,N};\ \sigma > 0)$, then the decision of system (4) and system (1) equivalent to it asymptotically is steady. (The proof of theorems 1 and 2 is carried out by the methods stated in work [2]).

Following generalized Theorem1, by means of replacement of equation (3) and equation (2), provide

$\dot{z} = (\Lambda_0 + \varepsilon N_1 + \underline{\underline{O}}(\varepsilon^3))z \equiv Q(t,\varepsilon)z,$

where $\Lambda_0 = \begin{pmatrix} \Lambda_{00} & 0 \\ 0 & \Lambda_{00} \end{pmatrix};\quad \Lambda_{00} = \begin{pmatrix} -i\omega & 0 \\ 0 & i\omega \end{pmatrix};\ N_1 = \begin{pmatrix} \overline{N}_{11} & \overline{N}_{12} \\ \overline{N}_{21} & \overline{N}_{22} \end{pmatrix};$

$\overline{N}_{11} = -\sigma\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \overline{N}_{22};\ \overline{N}_{12} = \frac{b_1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};\ \overline{N}_{21} = -\frac{b_1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

and the structure of a matrix of $Q(t,\varepsilon)$ allows to receive taking into account Theorem 2 having sufficient criteria of stability of initial system (1).

In the analysis of the nonlinear model system describing fluctuations of the thin ring resonator of a wave solid-state gyroscope with system of the sustaining torsions, it is necessary to use other constructive method in the theory of stability by using the method of unitary transformations.

Without damping it leads the ODE of the fourth order to system, which is mentioned as below

$$\dot{z} = \frac{\varepsilon}{8}\begin{pmatrix} 0 & -3\xi E & 4v - \xi X & 0 \\ 3\xi E & 0 & 0 & 4v - \xi X \\ -4v + \xi X & 0 & 0 & -3\xi E \\ 0 & -4v + \xi X & 3\xi E & 0 \end{pmatrix} z = \frac{\varepsilon}{8}Bz \tag{5}$$

where z is vector components, $z = (q_1,\ p_1,\ q_{2,}\ p_2)^T$; $q_1,\ p_1,\ q_2,p_2$- slowly changing variables connected with a form of fluctuations; ε - small parameter; ξ- the parameter characterizing nonlinear elasticity of material of the resonator; ν - dimensionless angular speed of the basis of a gyroscope; $E = q_1^2 + p_1^2 + q_2^2 + p_2^2$, $X = 2(p_2 q_1 - p_1 q_2)$ are the functions which are the first integrals of initial system.

As the matrix of **B** is normal ($\mathbf{B}^*\mathbf{B} = \mathbf{B}\mathbf{B}^*$, $\mathbf{B}^*$- the interfaced matrix) and skew symmetric and owing to this fact that, it has purely imaginary range [4], hence, we prove that the decision of equation (5) under any entry conditions are steady.

Theorem 3. For a square of standard of decisions of system

$$\dot{z} = A(t)z, \qquad z \in R^n \tag{6}$$

differential equality takes place $\frac{1}{2}\frac{d|z|^2}{dt} = Re(z^* A(t)z)$.

Proof: From equation (6), we get

$$\dot{z}^* = z^* A(t),\ z^* z = |z|^2$$

where $\frac{d|z|^2}{dt} = \dot{z}^* z + z^* \dot{z} = 2Re(z^* A(t)z)$

Theorem 4. If the matrix of $A(t)$ of equation (6) is identically normal having $(A^*(t)A(t) \equiv A(t)A^*(t))$ and its range $\{\lambda_j(t)\}_1^n$ satisfies $Re\lambda_j(t) \le \sigma(t)$, $\left(j = \overline{1,n};\ t \ge 0\right)$, then for a norm square the decision of equation (1) yield an inequalities: $\frac{d|z|^2}{dt} \le \sigma(t)|z|^2$,

$$|z(t)|^2 \le |z^0| e^{a(t)},\ a(t) = \int_0^t \sigma(\tau)\, d\tau.$$

Theorems 3 and 4 with equation (6) provide

$\frac{d|z|^2}{dt} \equiv 0$, guaranteeing existence of the steady decision.

For definition of more detailed structure we find the solutions of equation (6) with matrix **B** having range $\lambda_{1,2,3,4} = \pm i(a \pm b)$.

By means of unitary replacement of $z = \mathbf{U}x$, equation (6) is transformed to a type of $\dot{z} = \Lambda x$, where $\Lambda = diag\left(i(a+b), -i(a+b),\ i(a-b),\ -i(a-b)\right)$. Hence the common decision of initial equation (1) give

$$z = \mathbf{U}(\mathbf{C}_1 \cos(a+b)t + + \mathbf{C}_2 \sin(a+b)t + \mathbf{C}_3 \cos(a-b)t + +\mathbf{C}_4 \sin(a-b)t,$$

where $\mathbf{C}_{j,}$ $(j = \overline{1,4})$ - some constant vectors depending on entry conditions.

CONCLUSION

We find that splitting method is more efficient than method of unitary transformations. This motivates the search of new and more powerful integration methods for time independent partitioned Non Autonomous Equations, whose ultimate goal is to construct splitting methods showing a high efficiency both in autonomous and non-autonomous linear equations of the form (1) [2, 3] in the analysis of real model systems in the theory of gyroscopes.

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SURVIVAL FUNCTION OF FSH AND LH RESPONSES TO GNRH BY OVARIES STEROIDS IN LUTEAL PHASE OF THE CYCLE OF WOMEN

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ABSTRACT :
Aim of this paper is to find a mathematical model to determine the role of ovarian steroids in the control of GnRH induced gonadotrophin secretion in the luteal phase of the cycle of eighteen women subjects, by using survival function of Birnbaum- Saunders distribution. In reliability and survival analysis, it is often interest to determine the point at which the survival function reaches decreasing function of t from $-\infty$ to r_1, is an increasing function of t from r_1 to r_2 and is a decreasing function of t from r_2 to and is given by mathematical curves for the application part. It will be useful to medical professionals to find out the bounds of the corresponding curves for FSH and LH in different situations.
Keywords: ***Birnbaum-Saunders distribution, hazard function, survival function,Follicile stimulating hormone, Luteinizing hormone.***

1. MATHEMATICAL MODEL

1.1. Introduction

The two-Parameter Birnbaum-Saunders (BS) distribution was originally proposed by Birnbaum and Saunders[3] as a failure time distribution for fatigue failure caused under cyclic loading .The cumulative distribution function (CDF) of a two-Parameter BS random variable T is of the form

$$F(t;\alpha,\beta) = \Phi\left[\frac{1}{\alpha}\left\{\left(\frac{t}{\beta}\right)^{\frac{1}{2}} - \left(\frac{\beta}{t}\right)^{\frac{1}{2}}\right\}\right], \quad 0< t <\infty, \alpha, \beta> 0 \qquad (1)$$

Where $\Phi(.)$ is the standard normal CDF. The parameters α and β in (1) are the shape and scale parameters, respectively. Although the BS distribution was originally proposed as a failure time distribution for fatigue failure under the assumption that the failure is due to development and growth of a dominant crack, a more general derivation was provided by Desmond [4] based on a biological model. Desmond also strengthened the physical justification for the use of this distribution by relaxing the assumptions made originally by Birnbaum and saunders[3]. Some recent work on the BS distribution can be found in Balakrishnan et al.[2], and Ng et al. [10].

It is known from Johnson et al. [5] that the density function of the BS distribution is unimodal. In this paper, we first prove that the survival function of the BS distribution is decreasing function of t from $-\infty$ to r_1, is an increasing function of t from r_1 to r_2 and is a decreasing function of t from r_2 to ∞ for all values of the shape α and scale parameter β =1 which is used for our application part.

It is not uncommon to model survival and failure time data by distributions which have monotone hazard function. But in many practical situations, the hazard function is not monotone and in fact it increase up to a point and then decreases. For example, in the study of recovery from breast cancer, it has been observed by Langland's et al.[7] that the maximum mortality occurs after about three years and then it decreases slowly over a fixed period of time. Finally, here we have analysed a real data set and illustrate all the methods discussed.

1.2. Birnbaum- Saunders Distribution

The Probability density function (PDF) of a two – parameter BS random variable T corresponding to the CDF in (1) is given by

$$f(t;\alpha,\beta) = \frac{1}{2\sqrt{2\pi}\alpha\beta}\left[\left(\frac{\beta}{t}\right)^{\frac{1}{2}} + \left(\frac{\beta}{t}\right)^{\frac{3}{2}}\right] exp\left[-\frac{1}{2\alpha^2}\left(\frac{t}{\beta} + \frac{\beta}{t} - 2\right)\right],\ 0< t <\infty,\ \alpha, \beta> 0 \quad (2)$$

Consider now the monotone transformation

$$X = \frac{1}{2}\left[\left(\frac{T}{\beta}\right)^{\frac{1}{2}} - \left(\frac{T}{\beta}\right)^{-\frac{1}{2}}\right] \quad (3)$$

$$T = \beta\left\{1 + 2X^2 + 2X(1+X^2)^{\frac{1}{2}}\right\} \quad (4)$$

Then from (1), it readily follows that X is distributed as normal with mean zero and variance $(\frac{\alpha^2}{4})$. The transformation in (4) is a very useful transformation as it enables the determination of the moments of T through known results on expectations of functions of X.

1.3. Shape of the Hazard

To examine the shape of the hazard function, let us assume that the scale parameter β= 1,
without loss of generality. Let us consider the function $\in(t) = t^{\frac{1}{2}} - t^{-\frac{1}{2}}$

For which $\in^{'}(t) = \frac{d}{dt}\in(t) = \frac{1}{2}\left((t)^{-\frac{1}{2}} + (t)^{-\frac{3}{2}}\right) = \frac{1}{2t}\left(t^{\frac{1}{2}} + t^{-\frac{1}{2}}\right)$

$\in^{''}(t) = \frac{d}{dt}\in^{'}(t) = -\frac{1}{4t^2}\left(t^{\frac{1}{2}} + 3t^{-\frac{1}{2}}\right)$ and also $\in^2(t) = t + \frac{1}{t} - 2$

The density function of the BS distribution in (2) (for $\beta = 1$) is then

$$f(t;\alpha) = \frac{1}{\sqrt{2\pi}\alpha}\in^{'}(t)e^{-\frac{1}{2\alpha^2}\in^2(t)} \quad (5)$$

Which, in conjunction with the expression of the distribution function in (1), gives the hazard function as $h(t;\alpha) =$

$$\frac{f(t;\alpha)}{1-F(t;\alpha)} = \frac{\frac{1}{\sqrt{2\pi}\alpha}\in^{'}(t)e^{-\frac{1}{2\alpha^2}\in^2(t)}}{\phi\left(-\frac{\in(t)}{\alpha}\right)} \quad (6)$$

From (6), the shape of h(t, α) is not at all clear. We need the following lemmas for establishing our main result regarding the shape of the hazard function h(t,α) in (6).

Lemma1: Suppose f(t), for t> 0 is the density function of a positive real valued continuous random variable, $f^{'}(t)$ is the derivative of f(t) and η(t) = - $f^{'}(t)$/ f(t). Then, if there exist a t_0 such that $\eta^{'}(t) > 0\ \forall\ t\in(0, t_0)$, $\eta^{'}(t_0) = 0$ and $\eta^{'}(t) <0\ \forall\ t\in(t_0, \infty)$, the hazard function corresponding to f(t) is either an upside down of a decreasing function of t.

Lemma2: The hazard function of Birnbaum –Saunders distribution is either an upside down or a decreasing function of t> 0, for all values of the shape parameter α.

Proof: In this case, upon differentiating f(t,α) in (5) with respect to t, we get

$$f^{'}(t;\alpha) = \frac{1}{\sqrt{2\pi}\alpha}e^{-\frac{1}{2\alpha^2}\in^2(t)}\left(\in^{''}(t) - \frac{1}{\alpha^2}\left(\in^{'}(t)\right)^2\in(t)\right)$$

Consequently, we obtain

$$\eta(t,\alpha) = -\frac{f^{'}(t;\alpha)}{f(t;\alpha)} = -\left[\frac{\in^{''}(t)}{\in^{'}(t)} - \frac{\in(t)\in^{'}(t)}{\alpha^2}\right] = \frac{1}{2t} + \frac{1}{t(t+1)} + \frac{1}{2\alpha^2} - \frac{1}{2\alpha^2t^2}$$

And $\eta^{'}(t;\alpha) = \frac{S(t;\alpha)}{2(t+1)^2\alpha^2t^3}$

Where $S(t;\alpha) = -\alpha^2t^3 + (-6\alpha^2+2)t^2 + (-3\alpha^2+4)t + 2 \quad (7)$

Observe that S(0;α) = 2 and $S(\infty;\alpha) = \lim_{t\to1} S(t)$= -∞. It is clear that the signs of $S(t;\alpha)$ and η'(t; α) are the same ∀ t > 0. Let us consider the roots of

$$S^{'}(t;\infty) = -3\alpha^2t^2+2(-6\alpha^2+2)t+(-3\alpha^2+4) = 0 \quad (8)$$

Since the discriminant of the above quadratic equation

$4(27\alpha^4 - 12\alpha^2 + 4) = 4\{(3\alpha^2 - 2)^2 + 18\alpha^4\}$is always positive, both roots of the quadratic equation in (8) are real and are given by

$$\frac{-2(-6\alpha^2 + 2) \pm \sqrt{4(-6\alpha^2 + 2)^2 \ + 12\alpha^2(-3\alpha^2 + 4)}}{-6\alpha^2}$$

It can be easily seen that if $0 < \alpha < \frac{\sqrt{4}}{\sqrt{3}}$, one root is positive and the other one is negative. Let us done the two roots by r_1 and r_2, where $r_1 < r_2$.

Case1: $0 < \alpha < \frac{\sqrt{4}}{\sqrt{3}}$

In this case, $r_1 < 0$, $r_2 > 0$ and $s'(0;\alpha) = -3\alpha^2 + 4 > 0$. Hence, $s'(t;\alpha) < 0$ for $t < r_1$, $s'(t;\alpha) > 0$ for $r_1 < t < r_2$, and $s'(t;\alpha) < 0$ for $t > r_2$. This implies that $s(t;\alpha)$ is a decreasing function of t from $-\infty$ to r_1, is an increasing function of t from r_1 to r_2, and is a decreasing function of t again from r_2 to ∞. Now, upon using the facts that $s(0;\alpha)=2$ and $s(\infty;\alpha)=-\infty$, it readily follows that there exists a t_0 such that $s(t;\alpha) > 0 \ \forall \ t \in (0,t_0)$, $s(t_0;\alpha) = 0$ and $s(t;\alpha) < 0 \ \forall \ t > t_0$.

Case2: $0 \geq \frac{\sqrt{4}}{\sqrt{3}}$

In this case, by writing the second term of $s'(t;\alpha)$ in (8) as $2(-6\alpha^2+8)t-12t$, we see that $s'(t;\alpha) < 0$ for all $t > 0$. Hence, $s(t;\alpha)$ is a decreasing function of t for $t>0$ and it decreases from 2 to $-\infty$ which readily implies that there exists a $t_0 > 0$ such that $s(t;\alpha) > 0 \ \forall \ t \in (0,t_0)$, $s(t_0;\alpha)=0$, and $s(t;\alpha) < 0 \ \ \forall \ t > t_0$.

2. APPLICATION

2.1. Introduction

It has been established that ovarian steroids play an important role in the control of gonadotrophin secretion from the pituitary. Clinical experiments have shown that exogenous estrogen is able to suppress basal levels of LH and FSH during the follicular phase of the cycle (Messinis et al., 1992, [8]). On the other hand, changes in the production of endogenous estrogen, such as after ovarian stimulation with FSH or after bilateralovariectomy, result respectively in a decrease or increase of endogenous gonadotrophin values (Kamel et al., 1991, [6] ;Alexandris et al., 1997[1]). In the case of ovariectomy, the pattern of LH increase following the operation is similar to that of FSH, but the values for both gonadotrophin are persistently lower in women oophorectomized in the luteal rather than the follicular phase of the cycle [1].In-vivo experiments have shown marked changes in the responsiveness of the pituitary to GnRH during the normal menstrual cycle, with a significant increase from the early follicular phase to mid cycle and a progressive decline thereafter. In a recent study in women, we have demonstrated that following ovariectomy in the luteal phase of the cycle, the response of FSH to GnRH increased progressively, while that of LH declined markedly. This indicates a differential control of FSH and LH by the ovaries[1], but the mechanism is not clear.

The present study was undertaken to investigate the mechanism through which the ovaries control GnRH induced LH and FSH secretion during the luteal phase of the menstrual cycle by treating normal premenopausal women.

The study included 18 normally cycling women aged 42-46 years, with normal FSH values in the early follicular phase(<10IU/I). All women were studied during the week following bilateral ovariectomy plus total hysterectomy performed by laparotomy under general anaesthesia (09:00h). The ovaries were normal and the indications for the operation were benign uterine lesions, such as fibroids and menorrhagia. According to the group, which has been taken for this stud y (n= 6) no hormonal treatment was given to the women post-operatively.In women receiving hormonal treatment, contraindications for the administration of the steroids were identified.The operation was performed in the early to mid- luteal phase of the cycle, i.e. five days after the endogenous LH peak detected by LH measurement in daily blood samples taken from the time the follicle size was 16mm in diameter as assessed by ultrasound. In all women the pituitary response to GnRH (10μg.iv.) was investigated on a daily basis, starting in the morning before the operation until post- operative day 7, i.e. the day of discharge. Blood samples in relation to each GnRH injection (time0) were obtained at -15, 0, and 30 minutes. The 30 minute point was chosen because at that time a maximal response to GnRH has been reported and this represents pituitary sensitivity to GnRH.

During the operation, the presence of a corpusluteum was confirmed. Before the operation, all women have normal haemoglobin levels (> 12g/dl) and the operations were performed without any complications. The blood loss was < 300ml in all patients and the post- operative period was uneventful.

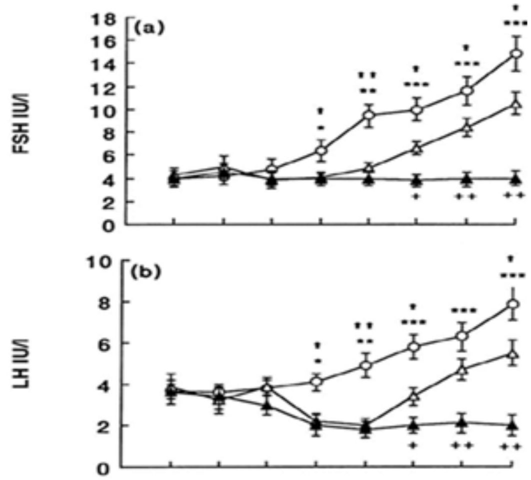


Figure2.1.1. Serum FSH , LH, values before and after bilateral ovariectomy plus hysterectomy performed in early to mid-luteal phase(day0) in 18 normally ovulating women. Six of the women (o) received no hormonal treatment post operatively (group 1).

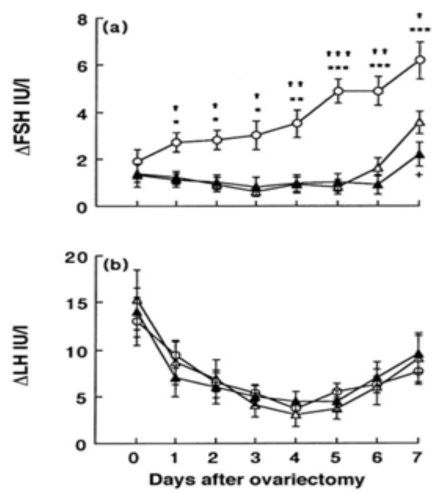


Figure2.1.2 Responses of FSH (ΔFSH) and LH (ΔLH) at 30 min to GnRH(10µg i.v) before and after bilateral ovariectomy plus hysterectomy performed in early to mid-luteal phase(day 0) in 18 normally ovulating women. Six of the women(o)received no hormonal treatment post operatively (group 1).

DISCUSSION:

In the present study, the increasing basal values of FSH and LH following ovariectomy in the women who did not receive hormonal treatment are in agreement with our previous data [1]. The greater increase in serum FSH values compared with LH is probably related to the lower metabolic clearance rate and higher production rate of FSH. In terms of changes in GnRH-induced FSH secretion in the untreated (control) group of women, the pattern was similar to that previously reported, i.e. a continuous rise following ovariectomy [1], thus illustrating a suppressing effect of the ovaries on the pituitary at that stage of the cycle. This confirms that the following ovariectomy, the pattern of changes in LH response to GnRH is different from that of FSH response.

The decreasing values of ΔLH in the women who did not receive hormonal treatment could be interpreted as indicating that the ovaries exerted a sensitizing effect on LH secretion before the operation. It is possible, therefore, that either a sensitizing effect of the ovaries on the pituitary is exerted through unspecified substances, or that the decrease in ΔLH values following ovariectomy is controlled by extra-ovarian mechanisms. Such mechanisms could be related to depleted stores of pituitary gonadotrophins as a result of the preceding mid-cycle LH surge that affected LH reserves more than those of FSH. The latter possibility is more likely based on previous data that a declining pattern of LH response to GnRH during the luteal phase of the cycle has also been reported in women with intact ovaries (Messinis et al., 1993, [9]). The fact, however, that following ovariectomy the decline in ΔLH was interrupted shortly after the operation, i.e. 4 days from the mid-luteal stage (Figure 2), while in women with intact ovaries the decline continues until the end of the luteal phase [9], indicates an earlier recovery of the pituitary in the ovariectomized than in the non-ovariectomized women. This suggests that GnRH-induced LH secretion in the luteal phase is not entirely unaffected by the ovaries. It is possible that a factor, maintains a lowresponsiveness of LH to GnRH towards the end of the cycle. Such a factor that specifically reduces LH response to GnRH is gonadotrophin surge attenuating factor (GnSAF) (Messinis and Templeton, 1989 [10]), but its role at that stage of the cycle needs to be further investigated.

3. MATHEMATICAL RESULT

Survivalfunction of FSH, LH are given in the figures2.1.1(a),2.1.1(b),2.1.2(a) and 2.1. 2(b) respectively by using the equation (7) are given in the following figures 3(a),3(b),3(c)and 3(d) respectively. All the curves are decreasing function of t from $-\infty$ to r_1, is an increasing function of t from r_1 to r_2 and is a decreasing function of t from r_2 to ∞.

Figure 3(a)

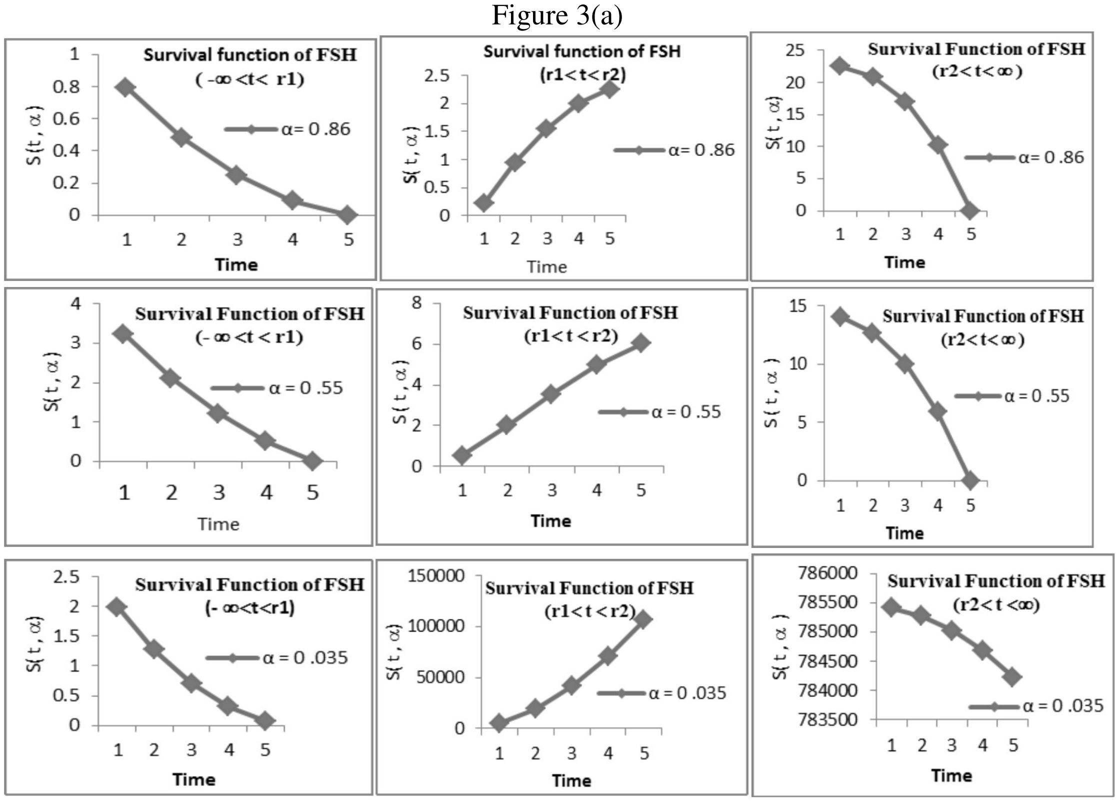


Figure 3(b)

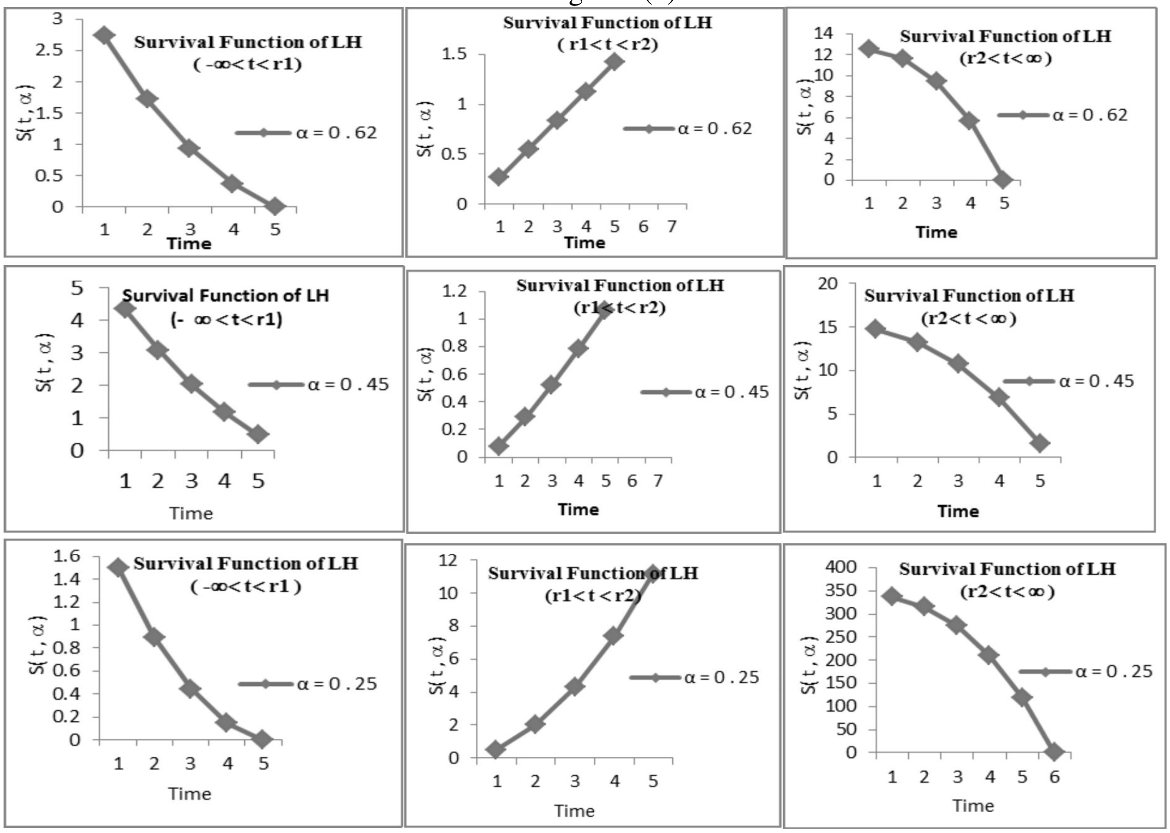


Figure 3(c):

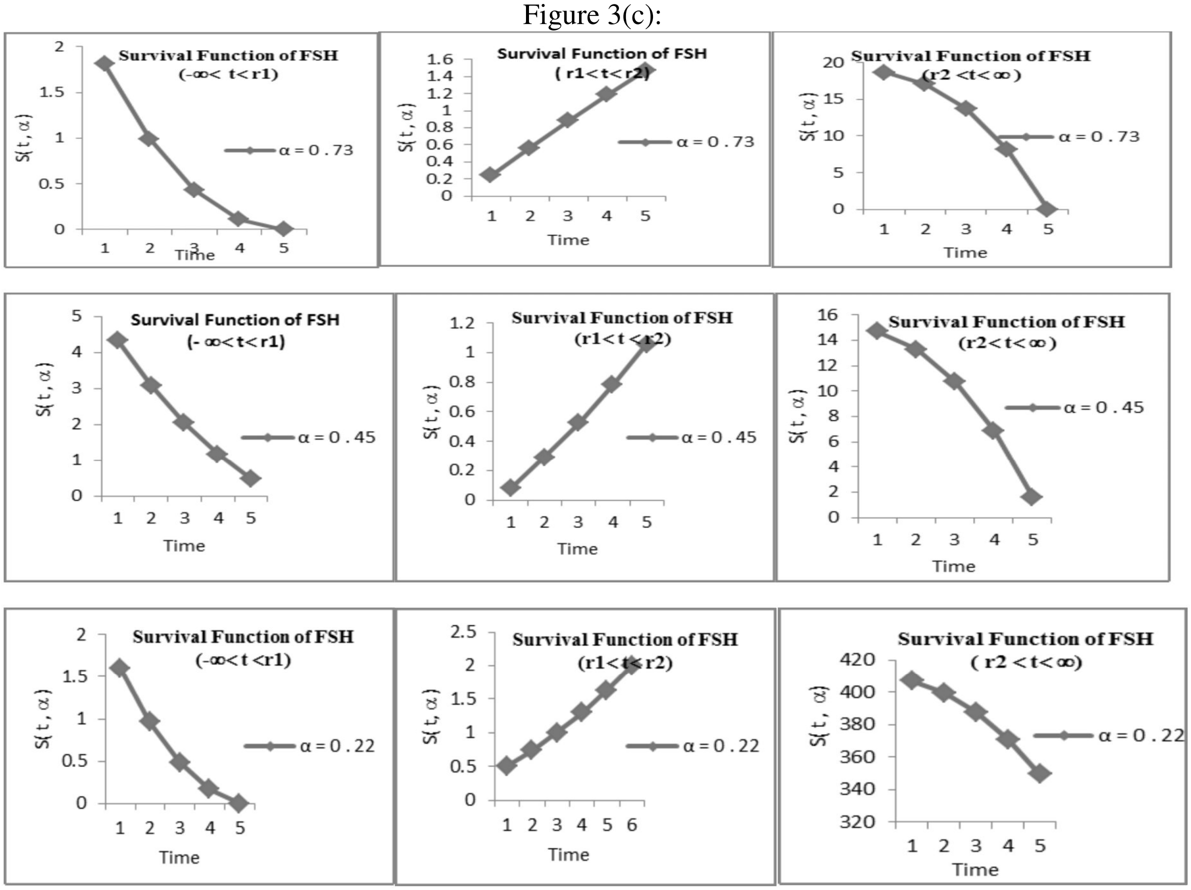
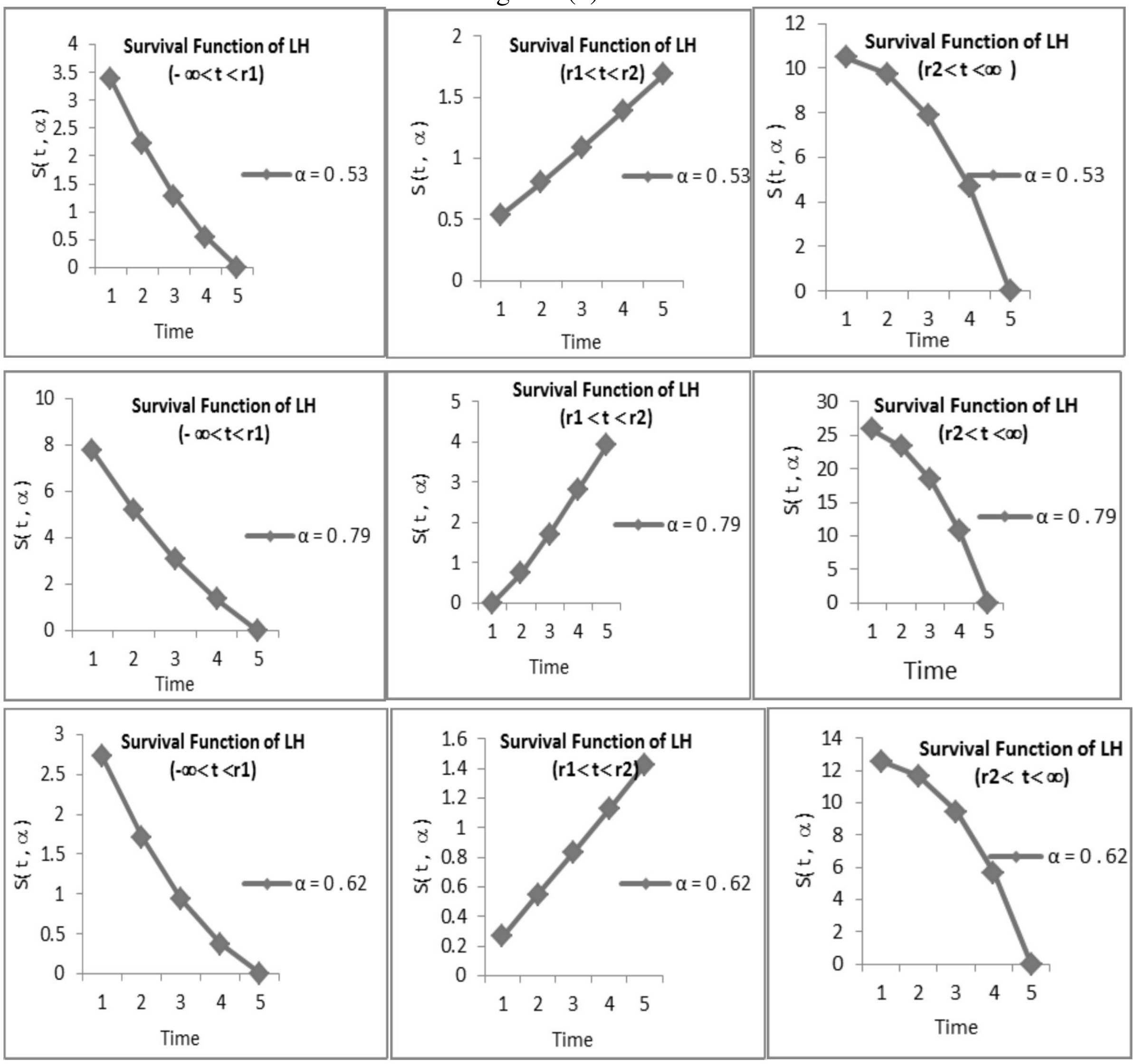


Figure 3(d):



4. CONCLUSION

In reliability and survival analysis, it is often used to determine the point at which the survival function reaches decreasing function of t from $-\infty$ to r_1, is an increasing function of t from r_1 to r_2 and is a decreasing function of t again from r_2 to ∞ and is given by the mathematical curves for the application part. It will be useful for the medical professionals to find out the bounds of the corresponding curves for FSH and LH in different situations. Survival function of FSH, LH is given in the figures 3(a), 3(b), 3(c) and 3(d) respectively. All the curves are decreasing function of t from $-\infty$ to $r_{1,}$is an increasing function of t from r_1 to $r_{2,}$ and is a decreasing function of t again from r_2 to ∞. It is conclude in the medical part that Δ FSH, Δ LH values show the same pattern of changes with a significant decline up to post-operative day 4 and a gradual increase thereafter.

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ECONOMIC ANALYSIS OF A RELIABILITY MODEL FOR TWO-UNIT COLD STANDBY SYSTEM WITH THREE TYPES OF REPAIR POLICY

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ABSTRACT :

The present paper analyses economic aspect of a reliability model for two-unit cold standby system with two types of repairman and three types of repair policy. On the failure of the unit, it is first undertaken by the ordinary repairman who may not be able to do some complex repairs and may rather damage the unit during try for its repair. On the inability shown by the ordinary repairman, the unit is undertaken for repair by the expert repairman adopting one of the three types of repair policy.

Various measures of system effectiveness have been obtained by making use of semi-Markov processes and regenerative point technique. Graphical study for a particular case is made and various cut-off points for making the decisions regarding profitability of the system have been obtained.

INTRODUCTION

Two–unit cold standby systems are also frequently used by various companies/industries/factories and hence have widely been studied by large number of researchers including [1-8] under various assumptions. Types of repair policy consider so far in the field of reliability were two in number, i.e., repeat repair policy and resume repair policy. But practically, there is also possibility of another type of repair policy which is the new concept investigated by the authors of the present paper. Earlier existed two types and presently introduced third type of repair policy are defined as follows :

(i) Resume repair policy

The repair of a failed unit is terminated before completion on the inability shown by ordinary repairman. When it begins by the expert, it is started from the stage where it was prior to the termination of repair.

(ii) Repeat repair policy (type I)

The repair of a failed unit is terminated due to incorrect process adopted by the ordinary repairman. Then, it begins by the expert repairman at the stage the ordinary repairman had taken over it.

(iii) Repeat repair policy (type II)

The repair of a failed unit is terminated due to mishandling or some other reasons by the ordinary repairman as a result of which unit gets damaged. Then the repair is begun by the expert repairman from the more degraded stage.

Taking into consideration the above practical situation, the present paper analyses a two-unit cold standby system with three types of repair policy. On the failure of the unit, it is undertaken by an ordinary repairman with the known fact that he may not be able to do some complex repairs. There may also be the possibility of rather damaging the unit by him during repair resulting it to go into more degraded stage. When the ordinary repairman finds himself unable, an expert repairman arrives who first discusses the process of repair done by the ordinary repairman. After discussion, if it is found that the process of earlier repair was correct and no mishandling occurred then resume repair policy is adopted whereas if the process was incorrect or there is some mishandling took place, one of the above mentioned types of repair policy is to be adopted. It is assumed that expert repairman repairs all the units which fail during his stay at the system. Other assumptions are as usual.

The system has been analysed by making use of semi–Markov processes and regenerative point technique. Various measures of system effectiveness have been obtained. Study through graphs is also made for a particular case.

NOTATIONS

λ	:	constant failure rate of an operative unit
p	:	probability that the ordinary repairman is able to complete the repair.
q	:	probability that the ordinary repairman is unable to complete the repair.
a	:	probability that the process of repair done by ordinary repairman was correct.
b_1	:	probability that the process of repair done by the ordinary repairman was incorrect but does not lead to any further damage.
b_2	:	probability that during the repair, the ordinary repairman leads the unit to more degraded stage.
h(t), h(t)	:	p.d.f., c.d.f of the discussion time.
g(t),G(t)	:	p.d.f., c.d.f. of repair time of the ordinary repairman.
$g_1(t),G_1(t)$	:	p.d.f., c.d.f. of repair time of the expert repairman when resume repair policy is adopted.
$g_2(t),G_2(t)$	:	p.d.f., c.d.f. of the repair time of the expert repairman when repeat repair policy (type I) is adopted.
$g_3(t),G_3(t)$	:	p.d.f., c.d.f. of the repair time of the expert repairman when repeat repair policy (type II) is adopted.

Symbols for the State of the System are :

o	:	Operative
cs	:	Cold standby
F_r	:	Failed unit under repair of the ordinary repairman.
F_R	:	Repair of the failed unit by the ordinary repairman is continuing from previous state.
F_{de}	:	Failed unit under discussion of the expert to know the nature of repair done by the ordinary repairman.
F_{De}	:	Failed unit waiting for repair while the discussion is continuing from the previous state.
F_{re_1}	:	Failed unit under repair of the expert repairman while resume repair policy is adopted.
F_{Re_1}	:	Repair of the failed unit by the expert repairman is continuing from the previous state under resume repair policy.
F_{re_2}	:	Failed unit under repair of the expert repairman while repeat repair policy (type I) has been adopted.
F_{Re_2}	:	Repair of the failed unit by the expert repairman is continuing from the previous state under repeat repair policy(type I).
F_{re_3}	:	Failed unit under repair of the expert repairman while repeat repair policy (type II) has been adopted.
F_{Re_3}	:	Repair of the failed unit by the expert repairman continuing from the previous state under repeat repair policy(type II).
F_w	:	Failed unit waiting for repair.

TRANSITION PROBABILITIES AND MEAN SOJOURN TIME

The transition diagram is shown as in **Fig. 1.** The epochs of entry into states 0, 1, 2, 4, 5, 6, 8, 12, 13, 14 are regeneration points and thus 0, 1, 2, 4, 5, 6, 8, 12, 13, 14 are regenerative states. States 3, 7, 8, 9, 10, 11, 12, 13, 14 are failed states.

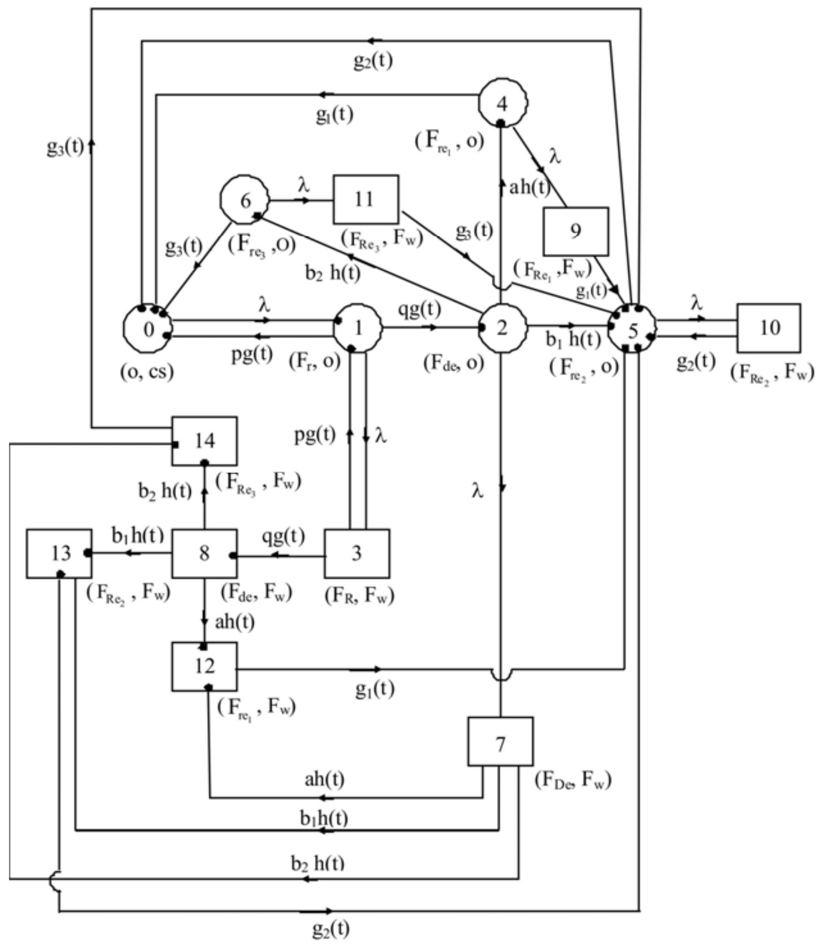

Fig. 1 State Transition Digram

Up state

Failed state

Regeneration Point

The non–zero elements $p_{ij} = \lim_{s\to 0} q_{ij}^*(s)$ are : –

$p_{01} = 1$, $p_{10} = pg^*(\lambda)$, $p_{12} = qg^*(\lambda)$

$p_{13} = 1 - g^*(\lambda)$, $p_{11}^{(3)} = p[1-g^*(\lambda)]$, $p_{18}^{(3)} = q[1-g^*(\lambda)]$

$p_{24} = a\,h^*(\lambda)$, $p_{25} = b_1h^*(\lambda)$, $p_{26} = b_2h^*(\lambda)$

$p_{27} = 1 - h^*(\lambda)$, $p_{2.12}^{(7)} = a[1-h^*(\lambda)]$, $p_{2.13}^{(7)} = b_1[1-h^*(\lambda)]$

$p_{2.14}^{(7)} = b_2[1-h^*(\lambda)]$, $p_{40} = g_1^*(\lambda)$, $p_{49} = 1 - g_1^*(\lambda)$

$p_{45}^{(9)} = 1 - g_1^*(\lambda)$, $p_{50} = g_2^*(\lambda)$, $p_{5,10} = 1 - g_2^*(\lambda)$

$p_{55}^{(10)} = 1 - g_2^*(\lambda)$, $p_{60} = g_3^*(\lambda)$, $p_{6,11} = 1 - g_3^*(\lambda)$

$p_{6,5}^{(11)} = 1 - g_3^*(\lambda)$, $p_{8,12} = a$, $p_{8,13} = b_1$

$p_{8,14} = b_2$, $p_{12,5} = 1$, $p_{13,5} = 1$, $p_{14,5} = 1$

…(1-28)

By these transition probabilities, it can be verified that

$p_{01} = 1, p_{10} + p_{12} + p_{13} = 1, \quad p_{10} + p_{12} + p_{11}^{(3)} + p_{18}^{(3)} = 1$

$p_{24} + p_{25} + p_{26} + p_{27} = 1, \quad p_{24} + p_{25} + p_{26} + p_{2,12}^{(7)} + p_{2,13}^{(7)} + p_{2,14}^{(7)} = 1$

$p_{40} + p_{49} = 1, \quad p_{40} + p_{45}^{(9)} = 1$

$p_{50} + p_{5,10} = 1, \quad p_{50} + p_{55}^{(10)} = 1$

$p_{60} + p_{6,11} = 1, \quad p_{60} + p_{65}^{(11)} = 1$ …(29-39)

The mean sojourn time (μ_i) in state i are :

$\mu_0 = \dfrac{1}{\lambda}, \quad \mu_1 = \dfrac{1-g^*(\lambda)}{\lambda}, \quad \mu_2 = \dfrac{1-h^*(\lambda)}{\lambda}$

$\mu_4 = \dfrac{1-g_1^*(\lambda)}{\lambda}, \quad \mu_5 = \dfrac{1-g_2^*(\lambda)}{\lambda}, \quad \mu_6 = \dfrac{1-g_3^*(\lambda)}{\lambda}$

$\mu_8 = -h^{*\prime}(0), \quad \mu_{12} = -g_1^{*\prime}(0), \quad \mu_{13} = -g_2^{*\prime}(0)$

$\mu_{14} = -g_3^{*\prime}(0)$ …(40-49)

The unconditional mean time taken by the system to transit for any state 'j', when it is counted from epoch of entrance into that state 'i' is mathematically stated as :

$$m_{ij} = \int_0^\infty t q_{ij}(t)\, dt = -q_{ij}^{*\prime}(0) \qquad \text{…(50)}$$

Therefore,

$m_{01} = \mu_0 \quad ; \quad m_{10} + m_{12} + m_{13} = \mu_1$

$m_{10} + m_{12} + m_{11}^{(3)} + m_{18}^{(3)} = \int_0^\infty t\, g(t)\, dt = k_1$ (say)

$m_{24} + m_{25} + m_{26} + m_{27} = \mu_2$

$m_{24} + m_{25} + m_{26} + m_{2.12}^{(7)} + m_{2,13}^{(7)} + m_{2,14}^{(7)} = \mu_8$

$m_{40} + m_{49} = \mu_4 \quad ; \quad m_{40} + m_{45}^{(9)} = \mu_{12}$

$m_{50} + m_{5,10} = \mu_5 \quad ; \quad m_{50} + m_{55}^{(10)} = \mu_{13}$

$m_{60} + m_{6,11} = \mu_6 \quad ; \quad m_{60} + m_{65}^{(11)} = \mu_{14}$

$m_{8,12} + m_{8,13} + m_{8,14} = \mu_8 \quad ; \quad m_{12,5} = \mu_{12}$

$m_{13,5} = \mu_{13} \quad ; \quad m_{14,5} = \mu_{14}$ …(51-65)

MEAN TIME TO SYSTEM FAILURE

By employing the arguments used for the regenerative process, we obtain the following recursive relations for $\phi_i(t)$:–

$\phi_0(t) = Q_{01}(t) \, Ⓢ \, \phi_1(t)$

$\phi_1(t) = Q_{10}(t) \, Ⓢ \, \phi_0(t) + Q_{12}(t) \, Ⓢ \, \phi_2(t) + Q_{13}(t)$

$\phi_2(t) = Q_{24}(t) \, Ⓢ \, \phi_4(t) + Q_{25}(t) \, Ⓢ \, \phi_5(t) + Q_{26}(t) \, Ⓢ \, \phi_6(t) + Q_{27}(t)$

$\phi_4(t) = Q_{40}(t) \, Ⓢ \, \phi_0(t) + Q_{49}(t)$

$\phi_5(t) = Q_{50}(t) \, Ⓢ \, \phi_0(t) + Q_{5,10}(t)$

$\phi_6(t) = Q_{60}(t) \, Ⓢ \, \phi_0(t) + Q_{6,11}(t)$ …(66-71)

Taking Laplace–Stieltjes Transforms (L.S.T.) of these relations and solving for $\phi_0^{**}(s)$. The mean time to system failure (MTSF) when the system starts from the state '0' is

$$T_0 = \lim_{s\to 0}\frac{1-\phi_0{}^{**}(s)}{s} = \frac{N}{D} \qquad \ldots(72)$$

where

$$N = \mu_0 + \mu_1 + p_{12}(\mu_2 + p_{24}\mu_4 + p_{25}\,\mu_5 + p_{26}\,\mu_6) \qquad \ldots(73)$$

$$D = p_{13} + p_{12}(1 - p_{24}\,p_{40} - p_{25}\,p_{50} - p_{26}\,p_{60}) \qquad \ldots(74)$$

AVAILABILITY ANALYSIS

Using the arguments of the theory of regenerative process, the availability $A_i(t)$ is seen to satisfy the following recursive relations :

$$A_0(t) = M_0(t) + q_{01}(t) © A_1(t)$$

$$A_1(t) = M_1(t) + q_{10}(t)©A_0(t) + q_{12}(t)©A_2(t) + q_{11}^{(3)}(t)©A_1(t)+q_{18}^{(3)}(t)©A_8(t)$$

$$A_2(t) = M_2(t) + q_{24}(t) © A_4(t) + q_{25}(t) © A_5(t) + q_{26}(t) © A_6(t) + q_{2,12}^{(7)}(t) © A_2(t) + q_{2,13}^{(7)}(t) © A_{13}(t) + q_{2,14}^{(7)}(t) © A_{14}(t)$$

$$A_4(t) = M_4(t) + q_{40}(t) © A_0(t) + q_{45}^{(9)}(t) © A_5(t)$$

$$A_5(t) = M_5(t) + q_{50}(t) © A_0(t) + q_{55}^{(10)}(t) © A_5(t)$$

$$A_6(t) = M_6(t) + q_{60}(t) © A_0(t) + q_{6,5}^{(11)}(t) © A_5(t)$$

$$A_8(t) = q_{8,12}(t) © A_{12}(t) + q_{8,13}(t) © A_{13}(t) + q_{8,14}(t) © A_{14}(t)$$

$$A_{12}(t) = q_{12,5}(t) © A_5(t)$$

$$A_{13}(t) = q_{13,5}(t) © A_5(t)$$

$$A_{14}(t) = q_{14,5}(t) © A_5(t) \qquad \ldots(75\text{-}84)$$

where

$$M_0(t) = e^{-\lambda t}, \quad M_1(t) = e^{-\lambda t}\,\overline{G}(t), \quad M_2(t) = e^{-\lambda t}\,\overline{H}(t)$$

$$M_4(t) = e^{-\lambda t}\,\overline{G}_1(t), \quad M_5(t) = e^{-\lambda t}\,\overline{G}_2(t), \quad M_6(t) = e^{-\lambda t}\,\overline{G}_3(t) \qquad \ldots(85\text{-}90)$$

Taking Laplace transforms of the above equations and solving them for $A_0^*(s)$, in steady-state, the availability of the system is given by

$$A_0 = \lim_{s\to 0}(s\,A_0{}^*(s)) = \frac{N_1}{D_1} \qquad \ldots(91)$$

where

$$N_1 = p_{50}[1 - p_{11}^{(3)}]\mu_0 + p_{50}\mu_1 + p_{12}p_{50}\,[\mu_2 + p_{24}\mu_4 + p_{26}\,\mu_6] + [p_{12}(1 - p_{24}\,p_{40} - p_{26}\,p_{60}) + p_{18}^{(3)}]\,\mu_5$$

$$D_1 = p_{50}[1 - p_{11}^{(3)}]\mu_0 + p_{50}k_1 + p_{50}[p_{12} + p_{18}^{(3)}]\mu_8 + p_{50}[p_{12}\,(p_{24} + p_{2,12}^{(7)}) + p_{18}^{(3)}p_{8,12}]\mu_{12} + [p_{12} + p_{18}^{(3)} + p_{12}\,p_{2,13}^{(7)}p_{50} + p_{18}^{(3)}\,p_{8,13}\,p_{50}]\mu_{13} + p_{50}[p_{12}\,(p_{26} + p_{2,14}^{(7)}) + p_{18}^{(3)}\,p_{8,14}]\mu_{14} \qquad \ldots(92\text{-}93)$$

BUSY PERIOD ANALYSIS OF THE ORDINARY REPAIRMAN

By probabilistic arguments, we have the following recursive relation for $B_i(t)$:

$$B_0(t) = q_{01}(t) © B_1(t)$$

$$B_1(t) = W_1(t)+q_{10}(t)©B_0(t) + q_{12}(t)©B_2(t) + q_{18}^{(3)}(t)©B_8(t) + q_{11}^{(3)}(t)©B_1(t)$$

$$B_2(t) = q_{24}(t) © B_4(t) + q_{25}(t) © B_5(t) + q_{26}(t) © B_6(t) + q_{2,12}^{(7)}(t) © B_{12}(t) + q_{2,13}^{(7)}(t) © B_{13}(t) + q_{2,14}^{(7)}(t) © B_{14}(t)$$

$$B_4(t) = q_{40}(t) © B_0(t) + q_{45}^{(9)}(t) © B_5(t)$$

$$B_5(t) = q_{50}(t) © B_0(t) + q_{55}^{(10)}(t) © B_5(t)$$

$B_6(t) = q_{60}(t) © B_0(t) + q_{65}^{(11)}(t) © B_5(t)$

$B_8(t) = q_{8,12}(t) © B_{12}(t) + q_{8,13}(t) © B_{13}(t) + q_{8,14}(t) © B_{14}(t)$

$B_{12}(t) = q_{12,5}(t) © B_5(t)$

$B_{13}(t) = q_{13,5}(t) © B_5(t)$

$B_{14}(t) = q_{14,5}(t) © B_5(t)$ …(94-103)

where

$W_1(t) = \overline{G}(t)$ …(104)

Taking L.T. of the above equations and solving them for $B_0^*(s)$, in steady-state, the total fraction of time for which the system is under repair of ordinary repairman is given by

$$B_0 = \lim_{s\to 0}(sB_0^*(s)) = \frac{N_2}{D_1} \quad …(105)$$

where

$N_2 = p_{50}\, k_1$ …(106)

and D_1 is already specified.

BUSY PERIOD ANALYSIS OF THE EXPERT REPAIRMAN (REPAIR TIME ONLY)

By probabilistic arguments, we have the following recursive relations for $B_i^e(t)$:

$B_0^e(t) = q_{01}(t) © B_1^e(t)$

$B_1^e(t) = q_{10}(t) © B_0^e(t) + q_{12}(t) © B_2^e(t) + q_{11}^{(3)}(t) © B_1^e(t) + q_{18}^{(3)}(t) © B_8^e(t)$

$B_2^e(t) = q_{24}(t) © B_4^e(t) + q_{25}(t) © B_5^e(t) + q_{26}(t) © B_6^e(t)$

$+ q_{2,12}^{(7)}(t) © B_{12}^e(t) + q_{2,13}^{(7)}(t) © B_{13}^e(t) + q_{2,14}^{(7)}(t) © B_{14}^e(t)$

$B_4^e(t) = W_4(t) + q_{40}(t) © B_0^e(t) + q_{45}^{(9)}(t) © B_5^e(t)$

$B_5^e(t) = W_5(t) + q_{50}(t) © B_0^e(t) + q_{55}^{(10)}(t) © B_5^e(t)$

$B_6^e(t) = W_6(t) + q_{60}(t) © B_0^e(t) + q_{65}^{(11)}(t) © B_5^e(t)$

$B_8^e(t) = q_{8,12}(t) © B_{12}^e(t) + q_{8,13}(t) © B_{13}^e(t) + q_{8,14}(t) © B_{14}^e(t)$

$B_{12}^e(t) = W_{12}(t) + q_{12,5}(t) © B_5^e(t)$

$B_{13}^e(t) = W_{13}(t) + q_{13,5}(t)\; B_5^e(t)$

$B_{14}^e(t) = W_{14}(t) + q_{14,5}(t) © B_5^e(t)$ …(107-116)

where

$W_4(t) = W_{12}(t) = \overline{G}_1(t)$

$W_5(t) = W_{13}(t) = \overline{G}_2(t)$

$W_6(t) = W_{14}(t) = \overline{G}_3(t)$

…(117-119)

Taking L.T. of the above equations and solving them for $B_0^e{}^*(s)$, in steady-state, the total fraction of the time for which the system is under repair of the expert repairman is given by

$$B_0^e = \lim_{s\to 0}(s B_0^e{}^*(s)) = \frac{N_3}{D_1} \quad …(120)$$

where

$N_3 = p_{50}[p_{12}\, p_{24} + p_{12}\; p_{2,12}^{(7)} + p_{18}^{(3)}\; p_{8,12}]\mu_{12} + [p_{12} – p_{12}\, p_{24}\, p_{40}$

$- p_{12}\, p_{26}\, p_{60} + p_{12}\, p_{2,13}^{(7)}\, p_{50} + p_{18}^{(3)} + p_{18}^{(3)}\, p_{8,13}\, p_{50}]\, \mu_{13} + [p_{12}\, p_{26}\, p_{50} + p_{18}^{(3)}\, p_{8,14}\, p_{50}]\mu_{14}$ …(121)

and D_1 is already specified.

EXPECTED DISCUSSION TIME

By probabilistic arguments, we have the following recursive relations for $DT_i(t)$:

$DT_0(t) = q_{01}(t) \copyright DT_1(t)$

$DT_1(t) = q_{10}(t) \copyright DT_0(t) + q_{12}(t) \copyright DT_2(t) + q_{11}^{(3)}(t) \copyright DT_1(t) + q_{18}^{(3)}(t) \copyright DT_8(t)$

$DT_2(t) = W_2(t) + q_{24}(t) \copyright DT_4(t) + q_{25}(t) \copyright DT_5(t) + q_{26}(t) \copyright DT_6(t)$
$\quad + q_{2,12}^{(7)}(t) \copyright DT_{12}(t) + q_{2,13}^{(7)}(t) \copyright DT_{13}(t) + q_{2,14}^{(7)}(t) \copyright DT_{14}(t)$

$DT_4(t) = q_{40}(t) \copyright DT_0(t) + q_{45}^{(9)}(t) \copyright DT_5(t)$

$DT_5(t) = q_{50}(t) \copyright DT_0(t) + q_{55}^{(10)}(t) \copyright DT_5(t)$

$DT_6(t) = q_{60}(t) \copyright DT_0(t) + q_{65}^{(11)}(t) \copyright DT_5(t)$

$DT_8(t) = W_8(t) + q_{8,12}(t) \copyright DT_{12}(t) + q_{8,13}(t) \copyright DT_{13}(t) + q_{8,14}(t) \copyright DT_{14}(t)$

$DT_{12}(t) = q_{12,5}(t) \copyright DT_5(t)$

$DT_{13}(t) = q_{13,5}(t) \copyright DT_5(t)$

$DT_{14}(t) = q_{14,5}(t) \copyright DT_5(t)$ …(122-131)

where

$W_2(t) = W_8(t) = \overline{H}(t)$ …(132)

Taking L.T. of the above equations and solving them for $DT_0^*(s)$, in steady-state, the total fraction of the time for which the expert is busy in discussion with ordinary repairman is given by

$$DT_0 = \lim_{s\to 0}(s\, DT_0^*\ (s)) = \frac{N_4}{D_1} \qquad \text{…(133)}$$

where

$N_4 = \mu_8\, p_{50}\, [p_{12} + p_{18}^{(3)}]$ …(134)

and D_1 is already specified.

EXPECTED NUMBER OF VISITS BY THE ORDINARY REPAIRMAN

By probabilistic arguments, we have the following recursive relations for $V_i(t)$:

$V_0(t) = Q_{01}(t) \circledS [1 + V_1(t)]$

$V_1(t) = Q_{10}(t) \circledS V_0(t) + Q_{12}(t) \circledS V_2(t) + Q_{11}^{(3)}(t) \circledS V_1(t) + Q_{18}^{(3)}(t) \circledS V_8(t)$

$V_2(t) = Q_{24}(t) \circledS V_4(t) + Q_{25}(t) \circledS V_5(t) + Q_{26}(t) \circledS V_6(t) + Q_{2,12}^{(7)}(t) \circledS V_{12}(t)$
$\quad + Q_{2,13}^{(7)}(t) \circledS V_{13}(t) + Q_{2,14}^{(7)}(t) \circledS V_{14}(t)$

$V_4(t) = Q_{40}(t) \circledS V_0(t) + Q_{45}^{(9)}(t) \circledS V_5(t)$

$V_5(t) = Q_{50}(t) \circledS V_0(t) + Q_{55}^{(10)}(t) \circledS V_5(t)$

$V_6(t) = Q_{60}(t) \circledS V_0(t) + Q_{65}^{(11)}(t) \circledS V_5(t)$

$V_8(t) = Q_{8,12}(t) \circledS V_{12}(t) + Q_{8,13}(t) \circledS V_{13}(t) + Q_{8,14}(t) \circledS V_{14}(t)$

$V_{12}(t) = Q_{12,5}(t) \circledS V_5(t)$

$V_{13}(t) = Q_{13,5}(t) \circledS V_5(t)$

$V_{14}(t) = Q_{14,5}(t) \circledS V_5(t)$ …(135-144)

Taking L.S.T. of the above equations and solving them for $V_0^{**}(s)$, in steady-state, the number of visits per unit time by the ordinary repairman is given by

$$V_0 = \lim_{t\to\infty}\left[\frac{V_0(t)}{t}\right] = \lim_{s\to 0}[sV_0 **(s)] = \frac{N_5}{D_1} \quad \text{...(145)}$$

where

$$N_5 = p_{50}\,[1 - p_{11}^{(3)}] \quad \text{...(146)}$$

and D_1 is already specified.

EXPECTED NUMBER OF VISITS BY THE EXPERT REPAIRMAN

By probabilistic arguments, we have the following relations for $V_i^e(t)$:

$$V_0^e(t) = Q_{01}(t) Ⓢ V_1^e(t)$$

$$V_1^e(t) = Q_{10}(t) Ⓢ V_0^e(t) + Q_{12}(t) Ⓢ [1 + V_2^e(t)] + Q_{11}^{(3)}(t) Ⓢ V_1^e(t) + Q_{18}^{(3)}(t) Ⓢ [1 + V_8^e(t)]$$

$$V_2^e(t) = Q_{24}(t) Ⓢ V_4^e(t) + Q_{25}(t) Ⓢ V_5^e(t) + Q_{26}(t) Ⓢ V_6^e(t) + Q_{2,12}^{(7)}(t) Ⓢ V_{12}^e(t) + Q_{2,13}^{(7)}(t) Ⓢ V_{13}^e(t) + Q_{2,14}^{(7)}(t) Ⓢ V_{14}^e(t)$$

$$V_4^e(t) = Q_{40}(t) Ⓢ V_0^e(t) + Q_{45}^{(9)}(t) Ⓢ V_5^e(t)$$

$$V_5^e(t) = Q_{50}(t) Ⓢ V_0^e(t) + Q_{55}^{(10)}(t) Ⓢ V_5^e(t)$$

$$V_6^e(t) = Q_{60}(t) Ⓢ V_0^e(t) + Q_{65}^{(11)}(t) Ⓢ V_5^e(t)$$

$$V_8^e(t) = Q_{8,12}(t) Ⓢ V_{12}^e(t) + Q_{8,13}(t) Ⓢ V_{13}^e(t) + Q_{8,14}(t) Ⓢ V_{14}^e(t)$$

$$V_{12}^e(t) = Q_{12,5}(t) Ⓢ V_5^e(t)$$

$$V_{13}^e(t) = Q_{13,5}(t) Ⓢ V_5^e(t)$$

$$V_{14}^e(t) = Q_{14,5}(t) Ⓢ V_5^e(t) \quad \text{...(147-156)}$$

Taking L.S.T. of the above equations and solving them for $V_0^e **(s)$, in steady-state, the number of visits per unit time by the expert repairman is given by

$$V_0^e = \lim_{s\to 0}[sV_0^e **(s)] = \frac{N_6}{D_1} \quad \text{...(157)}$$

where

$$N_6 = p_{50}[p_{12} + p_{18}^{(3)}] \quad \text{...(158)}$$

and D_1 is already specified.

PROFIT ANALYSIS

The expected total profit in steady-state is

$$P = C_0A_0 - C_1B_0 - C_2 B_0^e - C_3DT_0 - C_4V_0 - C_5 V_0^e \quad \text{...(159)}$$

where

C_0 = revenue per unit up time of the system
C_1 = cost per unit time for which ordinary repairman is busy
C_2 = cost per unit time for which expert repairman is busy in repairing the unit
C_3 = cost per unit time for which expert repairman is busy in discussion with ordinary repairman.
C_4 = cost per visit of the ordinary repairman
C_5 = cost per visit of the expert repairman

PARTICULAR CASE

For graphical interpretation, the following particular case is considered :

$$g(t) = \alpha e^{-\alpha t} \quad ; \quad g_1(t) = \alpha_1 e^{-\alpha_1 t}$$

$g_2(t) = \alpha_2 e^{-\alpha_2 t}$; $g_3(t) = \alpha_3 e^{-\alpha_3 t}$

$h(t) = \beta e^{-\beta t}$ …(160-164)

On the basis of the numerical values taken as :

$p = 0.5,\ q = 0.5,\ a = 0.2,\ b_1 = 0.7,\ b_2 = 0.1,\ \beta = 10,$

$\alpha = 0.25,\ \alpha_1 = 0.4,\ \alpha_2 = 0.35,\ \alpha_3 = 0.2,\ \lambda = 0.05$

The values of various measures of system effectiveness are obtained as :

Mean Time to System Failure (MTSF) = 118.27413

Availability (A_0) = 0.9323

Busy period of the ordinary repairman (B_0) = 0.5353278

Busy period of the expert repairman (B_0^e) = 0.316988

Expected discussion time (DT_0) = 0.005417

Expected number of visits by the ordinary repairman (V_0) = 0.106747

Expected number of visits by the expert repairman (V_0^e) = 0.05417

GRAPHICAL INTERPRETATION

The above particular case is considered for the graphical interpretation.

Fig. 2 shows the behaviour of profit with respect to failure rate for different values of repair rate (α). The profit decreases as the failure rate increases and is higher for higher value of repair rate (α). The following inferences can be made :

For $\alpha = 0.2$, 0.25 and 0.3, the system is profitable only if $\lambda < 0.0708$, 0.0801 and 0.087 respectively.

So, the companies using such systems can be suggested to purchase only those systems which do not have failure rates greater than those mentioned above.

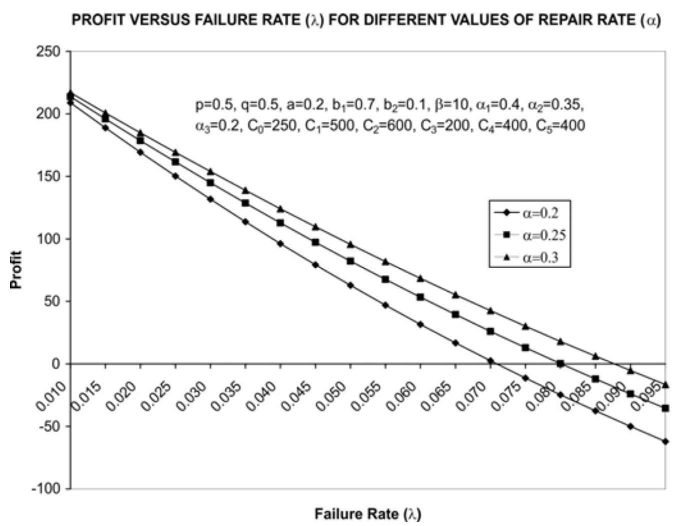

Fig. 2

Fig. 3 shows the behaviour of profit with respect to revenue per unit time (C_0) for different values of cost (C_2). The profit increases as C_0 increases and becomes lower for higher values of C_2. Following conclusions are drawn from this figure :

For $C_2 = 2000$, 4000 and 6000, the system is profitable only if $C_0 > 182.91$, 255.85 and 328.85 respectively.

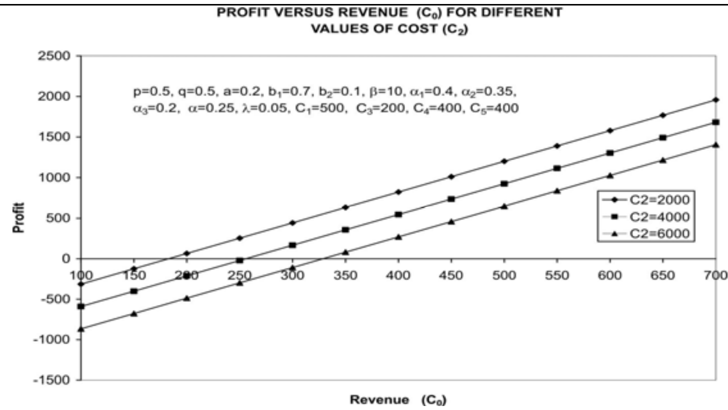

Fig. 3

Fig. 4 depicts that the profit increases with increase in the values of probability (p). Profit is higher for higher values of probability (a). It is noticeable that

For a = 0.1, 0.4 and 0.7, the system is profitable only if p > 0.328, 0.305 and 0.28 respectively.

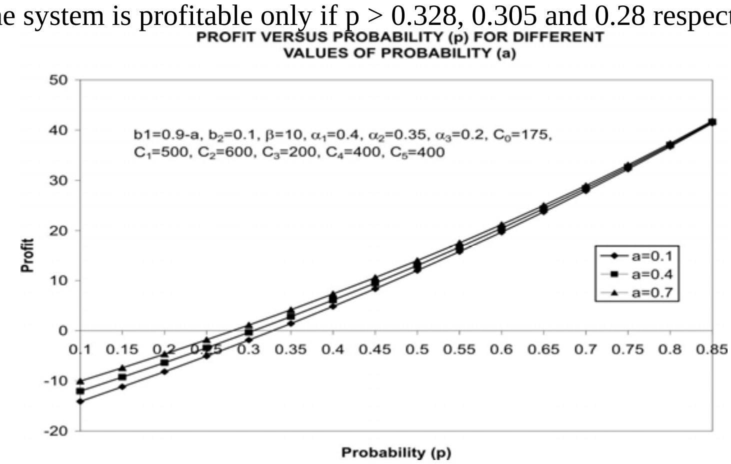

Fig. 4

It is also observed that the three curves converge as p → 1 which implies that profit comes out to be same as p → 1 irrespective of the values of probability (a).

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A FUZZY GOAL PROGRAMMING APPROACH FOR ACHIEVING SUSTAINABILITY IN CONSTRUCTION INDUSTRY

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ABSTRACT :
Pollution in basic sustainable resources due to xenobiotic activities is a serious problem today and construction industry is a major contributor to the increase of this problem. Rapidly increasing population and luxurious desires of people has caused a boom in construction industry and a rapid degradation in environment quality. Construction industry is a broad spectrum pollution causing industry which pollutes almost every resource like soil, water, air, etc which is a serious environmental issue.

In this paper we have made an attempt to present efficiency of Fuzzy Goal Programming (FGP) in the area of mathematical programming for modeling and solving environmental hazards due to construction industry.
Keyords: ***Fuzzy Goal Programming, xenobiotic activities, sustainability, pollution.***

INTRODUCTION:

With rapid increase in industrialization and uncontrolled population explosion a boom in construction industry has been noticed in last 1.5 decade. Approximately 10% of the GDP comes from construction industry and it has been growing at 10% annually during last decade. It is estimated that built up area will increase to approximately five times by the year 2030.

People are increasing and also their luxuries are increasing. This increase is affecting construction industry positively. Construction of roads, hotels, schools, luxurious apartments, etc is increasing day by day. This massive construction is regarded as sign of development in today's era as these help to meet luxurious desires of people and also is a huge employment providing sector. Construction industry is supported by many primary and secondary industries like mining, cement and brick manufacturing, iron and steel, etc. thus it is an important industry for economic development of the nation.

Apart from all these facts, environmental pollution and ecosystem disruption are also inseparable factors of this industry. No one can deny from the fact that construction industry is the major pollution causing sector in the world. It is estimated that construction industry contributes to about 23% of air pollution, 50% of the climate change, 40% water pollution, 50% landfill waste and 50% ozone depletion. Resource depletion, waste production and recycling are other major impacts of construction industry. Construction and subordinate activities like extraction of resources, burning of fossil fuels, etc. cause loss of habitat and ecosystem, damage to landscape and release of harmful gases. Presence of hazardous substances like sewage, V.O.C, T.S.P.M, and other wastes is the other major impact of construction industry.

Each and every construction may be small or big leaves a permanent and irreparable environmental impact. With the large scale increase in construction projects the environmental and health hazards are also increasing. Today we are standing on the cross roads of development and ecosystem conservation but a large part of population living on this planet choose development at the cost of life of the ecosystem. Now a-days the concept of sustainable development has been generated but it is still in its infancy in our country.

In this paper, we have tried to reveal the real condition of pollution by construction industry and use of a Goal programming model to control it.

GOAL PROGRAMMING:

Goal programming (GP) is a multi-objectives analytical approach devised to address decision-marking problems where targets have been assigned to all attributes and where the decision makers (DMs) are interested in minimizing the non-achievement of the corresponding goal. The model allows taking into account simultaneously many objectives while the decision-marking is seeking the best solution from among a set of feasible solutions. GP was first introduced by Charnes and Cooper (1961), and further developed by Lee (1972), Ignizio (1976), Tamiz, Jones, and Romero (1998), Romero (2001), Chang (2004); among others. The oldest form can be expressed as follows:

(GP model)

Minimize $\sum \left| f_i (X) - g_i \right|$ subject to

$X \varepsilon F$. F is a feasible set.

Where $f_i(X)$ is the linear function of the i^{th} goal, g_i is the aspiration level of the i^{th} goal.

[ref. Formulating the multi-segment goal programming, Chin-Nung Liao]

Charnes and cooper [A. Charnes and W. W. Cooper. Management models and industrial applications of linear programming. Vol 1. (Wiley, New York, 1961)] have introduced the concept of GP to solve the unsolvable linear programming problems.

FUZZY GOAL PROGRAMMING:

The fuzzy set theory is recurrently used in recent research. A fuzzy set A can be characterized by a membership function, which assigns to each object of a domain its grade of membership in A (L.A. Zadeh). The more an element or an object can be said to belong to a fuzzy set A, the closer to 1 is its grade of membership. Various types of membership functions can be used to support the fuzzy analytical framework although the fuzzy description is hypothetical and membership values are subjective (M. Belmokaddem).

A Fuzzy Goal Programming approach for linear programming problems with several objectives was developed by Zimmermann (1978). Narasimhan, (1980) were the first to give a Fuzzy Goal Programming formulation by using the concept of membership functions. These functions are defined on the interval [0, 1]. So the membership function for the i^{th} goal has the value of 1 when this goal is attained and the decision multi criteria is totally satisfied; otherwise the membership function assumes a value between 0 and 1.

$$f_k(x) = Aspiration\ level\ assigned\ to\ k^{th}\ objective$$

$$k = 1, 2, \ldots\ldots\ldots, k$$

Fuzzy Goal may appear as one of the following forms:

$$f_k(x) > A_k$$

$$f_k(x) = A_k$$

$$f_k(x) < A_k$$

x is the vector of decision variables.

For >

$$\mu_k(X) = \begin{cases} 1, & f_k(x) \geq A_k \\ \dfrac{f_k(x) - (A_k - L_k)}{L_k}, & A_k - L_k \leq f_k(x) < A_k \\ 0, & f_k(x) < A_k - L_k \end{cases}$$

For <

$$\mu_k(X) = \begin{cases} 1, & f_k(x) \le A_k \\ \dfrac{(A_k + U_k) - f_k(x)}{U_k}, & A_k \le f_k(x) < A_k + U_k \\ 0, & f_k(x) > A_k + U_k \end{cases}$$

For =

$$\mu_k(X) = \begin{cases} 1, & f_k(x) = A_k \\ \dfrac{(A_k + U_k) - f_k(x)}{U_k}, & A_k < f_k(x) < A_k + U_k \\ \dfrac{f_k(x) - (A_k - L_k)}{L_k}, & A_k - L_k \le f_k(x) < A_k \\ 0, & f_k(x) > A_k + U_k \text{ or } f_k(x) < A_k - L_k \end{cases}$$

CASE STUDY:

The study is based on pollution created by construction industry in our sustainable resources as noise, air pollutants and water pollutants. The pollutants considered for the study are:

1. **Noise:** It is an interesting fact that voice which gives sensation of hearing and through which we percept about 11% of the total stimulus around us when crosses a certain limit can even permanently damage the ability to hear. Now a-days due to frequent and irrational use of heavy machines noise pollution has become a major problem to the residents living around construction sites. Noise pollution has following effects on human beings:

a) Irritability
b) Loss of patience
c) Temporary hearing loss
d) Nausea
e) Permanent hearing loss

The various sources of noise pollution and their intensity at 50m from the construction site are given as:

Source	Noise Level (decibels)
1. Air compressor	81
2. 110 kvA diesel generator	95
3. Concrete Mixer	85
4. Milling Machine	112
5. Oxy- acetylene cutting	96
6. Pulverizer	92
7. Rivetting	95
8. Power operated portable saw	108
9. Steam Turbine	91
10. Pneumatic chiesling	105
11. Truck horns	100
12. Car horns	95
13. Jack Hammer	90
14. Pile Driver	102
15. Compact pump	82

Table 1: Source Manual by CPCB

2. **Total Suspended Particulate Matter(TSPM):** These in ambient air is a complex multiphase system consisting of particle sizes ranging from less than 0.01 µm to more than 100 µm. (wan kun et. Al. 2006; Devi, dahiya, Gadgil, singh, kumar, 2007). It is a quantitative pollutant which is produced from soil, plants and different human activities in different forms. Construction industry is a major producer of TSPM as various activities like cutting, grinding, chiseling, digging soil, throwing wastes of cement and bricks etc all add TSPM to the ambient air. This pollutant enters the respiratory system and causes various respiratory ailments like asthma, allergy, etc.
3. **Dissolved Solids, Oil, Grease, Carbonates and Iron:** There are various dissolved solids in water effluent from the construction sites which is being discharged either directly or indirectly to the nearby water bodies and polluting the water bodies. These include various heavy metals, metal oxides, sulphates, sulphides, chlorides, carbonates, silica etc. These metals enter the food chain when this polluted water is consumed directly or used for irrigation purpose. It is accumulated in the vital parts like kidney and liver in the body and cause various fatal diseases. The various activities of the construction sites responsible for these effluents are paints and varnishes, cement, various chemicals, cutting, grinding and polishing of granite and marble.

CPCB Guidelines: The guidelines given in the manual of CPCB for the maximum amount of various pollutants are as:

S. No	Land Use	8 hours L_{eq} (db)		30 days' average L_{dn} (db)
		Day	Night	
1	Residential	80	70	75
2	Commercial	85	85	80
3	Industrial	90	90	85

Table2: permissible level of noise. Source CPCB

Area / Pollutant	Industrial Area	Residential Area	Sensitive Area
TSPM	360 µg / m^3	140 µg / m^3	70 µg / m^3

Table3: permissible level for TSPM. Source CPCB

S. No.	Name of Pollutant	Maximum Permissible Limit (mg/ lit.)
1.	Dissolved Solids	2100
2.	Oil and Grease	10
3.	Carbonates	500
4.	Iron	0.3

Table 4: permissible level for various pollutants. Source CPCB

For the study three residential sites were selected and data was collected for noise, TSPM, dissolved solids, oil and grease, carbonates and iron from various effluents of the sites. The readings obtained are as:

S. No.	Site	Noise (db)	TSPM (mg/ m^3)	Dissolved Solids (mg/ lit)	Oil and grease (mg/lit)	Carbonates (mg/lit)	Irom (mg/lit)
1	I	105.006	860	2330	14.7	657.4	0.82
2	II	107.981	940	2475	15.3	759.2	0.74
3	III	118.052	895	2560	14.2	578.3	0.84

Table 5: Data of various pollutants from 3 different construction sites.

FUZZY GOAL PROGRAMMING MODEL:

The membership function for the above said problem is determined as:

Function 1: Minimize the total sound produced by various machines.

$$u_{z1} = \begin{cases} 1, & \text{if } z_1 \le 70. \\ \dfrac{105.006 - z_1}{10.006 - 70}, & \text{if } 70 \le z_1 \le 105.006 \\ 0, & \text{if } z_1 \ge 105.006 \end{cases}$$

Function 2: Minimize the TSPM produced by various activities on a construction site.

$$u_{z2} = \begin{cases} 1, & \text{if } z_2 \le 140. \\ \dfrac{860 - z_2}{10.006 - 70}, & \text{if } 140 \le z_2 \le 860 \\ 0, & \text{if } z_2 \ge 860 \end{cases}$$

Function 3: Minimize the amount of dissolved solids in effluents from a construction site.

$$u_{z3} = \begin{cases} 1, & \text{if } z_3 \le 2100. \\ \dfrac{2330 - z_3}{2330 - 2100} & \text{if } 2100 \le z_3 \le 2330 \\ 0, & \text{if } z_3 \ge 2330 \end{cases}$$

Function 4: Minimize the amount of oil and grease from effluents of a construction site.

$$u_{z4} = \begin{cases} 1, & \text{if } z_4 \le 10. \\ \dfrac{14.2 - z_4}{14.2 - 10}, & \text{if } 10 \le z_4 \le 14.2 \\ 0, & \text{if } z_4 \ge 14.2 \end{cases}$$

Function 5: Minimize the amount of carbonates from effluents of a construction site.

$$u_{z5} = \begin{cases} 1, & \text{if } z_5 \le 500. \\ \dfrac{578.3 - z_5}{578.3 - 500}, & \text{if } 500 \le z_5 \le 578.3 \\ 0, & \text{if } z_5 \ge 578.3 \end{cases}$$

Function 6: Minimize the amount of iron from effluents of a construction site.

$$u_{z6} = \begin{cases} 1, & if\ z_6 \le 0.30 \\ \dfrac{0.74 - z_6}{0.74 - 0.30}, & if\ 0.74 \le z_6 \le 0.30 \\ 0, & if\ z_6 \ge 0.74 \end{cases}$$

The objective functions are:

$$Max.\, f(u) = u_1 + u_2 + u_3 + u_4 + u_5 + u_6$$

Subject to constraints:

$$u_1 = \frac{105.006 - z_1}{10.006 - 70};$$

$$u_2 = \frac{860 - z_2}{860 - 140};$$

$$u_3 = \frac{2330 - z_3}{2330 - 2100};$$

$$u_4 = \frac{14.2 - z_4}{14.2 - 10};$$

$$u_5 = \frac{578.3 - z_5}{578.3 - 500};$$

$$u_6 = \frac{0.74 - z_6}{0.74 - 0.30};$$

RESULTS:

The formulated equations were solved using LINGO software and following results were obtained:

Pollutant	Original value	Optimized value	Result
Noise level (db)	105.006	68.96	Achieved
TSPM (mg/m^3)	860	194	Achieved
Dissolved Solids (mg/ lit)	2330	2500	Not Achieved
Oil and Grease (mg/ lit)	14.2	1.601	Achieved
Carbonates (mg/ lit)	578.3	56.91	Achieved
Iron (mg/ lit)	0.30	0.30	Achieved

From the above table we can clearly see that 5 out of 6 constraints have been achieved considerably. The noise level has been reduced by 34.3% and has been reduced to level less than permissible limits. The TSPM level has been reduced by 77.4% which is very high value and helpful to avoid air pollution. There is a considerable change in the oil and grease coming out as effluents from a construction site which is helpful in achieving good quality water. The carbonates in the effluents has been reduced by approximately 90% which is a high quantity in this era of pollution and very low than the permitted quantity. The quantity of iron has been reduced exactly up to the permitted level. The amount of dissolved solids has not been achieved by the model.

CONCLUSION:

The problem was solved using LINGO and five out of six objectives were achieved. The quantity of dissolved solids remained above government standards and hence remained unachieved. The results shown in this work shows in a multiple objective problem it is very difficult rather impossible to reach all the goals, but reaching an optimal solution is feasible and more realistic. The results shown in this work shows in a multiple objective problem it is very difficult rather impossible to reach all the goals, but reaching an optimal solution is feasible and more realistic. Thus, multi- objective goal programming can be a useful tool in optimizing pollution of sound, air and water from various construction sites both residential and industrial sites and thus helpful in conservation of environment and human health. This is a basic study and thus there is a scope of changes and modifications in it.

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A NEW APPROACH TO SOLVE FULLY FUZZY TRANSPORTATION PROBLEM

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ABSTRACT :
In the literature, there are various method to solve transportation problem in which some or all parameters are represented by triangular or trapezoidal fuzzy numbers. In this paper, we solve the fully fuzzy transportation problem using fuzzy version of VAM and MODI algorithms and the Operations for Subtraction and Division on Triangular Fuzzy Number the advantage of these operations is to undergo the inverse operations of the addition and multiplication and then compared between our proposed method and existing methods. Numerical problem is also provided to demonstrate the effectiveness of proposed method.

Keywords: ***Fuzzy Arithmetic, Fuzzy Set, Fuzzy Transportation Problem, Triangular Fuzzy Number.***

INTRODUCTION

The basic transportation problem was originally developed by Hitchcock (1941). Charnes and Cooper (1954) developed the stepping stone method which provides an alternative way of determining the simplex method information. Dantzig and Thapa (1963) used the simplex method to the transportation problem as the primal simplex transportation method. An initial basic feasible solution for the transportation problem can be obtained by using the north-west corner rule, row minima, column minima, matrix minima (least-cost), or the vogel's approximation method. The modified distribution method is useful for finding the optimal solution for the transportation problem.

In conventional transportation problems it is assumed that decision maker is sure about the precise values of transportation cost, availability and demand of the product. In real world applications, all the parameters of the transportation problems may not be known precisely due to uncontrollable factors. This type of imprecise data is not always well represented by random variable selected from a probability distribution. Fuzzy numbers may represent these type of data. So, fuzzy decision making method is needed here.

Zimmermann (1978) showed that solutions obtained by FLP are always efficient. Subsequently, Zimmermann's FLP has developed into several fuzzy optimization methods for solving the transportation problem. Oheigeartaigh (1982) proposed an algorithm for solving transportation problems where the capacities and requirements are fuzzy sets with linear or triangular membership functions. Chanas et al. (1984) presented a FLP model for solving transportation problems with crisp cost coefficients and fuzzy supply and demand values. Chanas and Kuchta (1996) proposed the concept of the optimal solution for the transportation problems with fuzzy coefficient expressed as fuzzy number, and developed an algorithm for obtaining for optimal solution. Saad and Abbas (2003) discussed the solution algorithm for solving the transportation problems in fuzzy environment.

Liu and Kao (2004) described a method for solving fuzzy transportation problems based on extension principle. Gani and Razak (2006) presented a two stage cost minimizing fuzzy transportation problems in which supplies and demands are trapezoidal fuzzy numbers. A parametric approach is used to obtain a fuzzy solution and the aim is to minimize the sum of the transportation cost in the two stages. To deal with uncertainties of supply and demand parameters, Gupta and Mehlawat (2007) transformed the past data pertaining to the amount of supply of the ith supply point and the amount of demand of the jth demand point using level (λ , p) interval valued fuzzy numbers.

Dinagar and Palanival (2009) investigated fuzzy transportation problem, with the aid of trapezoidal fuzzy numbers and proposed fuzzy modified distribution to find the optimal solution in terms of fuzzy numbers. Pandian and Natarajan (2010) proposed a new algorithm namely, fuzzy zero point method for finding a fuzzy optimal solution for a fuzzy transportation problems, where the transportation cost, supply and demand are represented by trapezoidal fuzzy numbers.

Kaur and Kumar (2011) proposed a new method for solving fuzzy transportation problem by assuming that a decision maker is uncertain about the precise values of the transportation cost, availability and demand of the product. Narayanamoorthy et al. (2013) proposed Fuzzy Russell's method to find the initial basic feasible solution using Yager's ranking method with trapezoidal fuzzy numbers. Antony et al. (2014) proposed a Method for Solving the Transportation Problem Using Triangular Intuitionistic Fuzzy Number. Shanmugasundari and Ganesan [2] proposed a fuzzy version of VAM and MODI algorithms to solve fuzzy transportation problem without converting them to classical transportation problem. In this paper, we solve the fully fuzzy transportation problem using Fuzzy version of Vogels and Modi algorithms proposed by Shanmugasundari and Ganesan (2013) and the Operations for Subtraction and Division on Triangular Fuzzy Number proposed by Gani and Assarudin(2012).

2. PRELIMINARIES

Fuzzy set (Kumar and Bhatia 2011): A fuzzy set $\tilde{A}$ in X (set of real number) is a set of ordered pairs:

$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) / x \in X\}$ $\mu_{\tilde{A}}(x)$ is called membership function of x in $\tilde{A}$ which maps X to [0,1].

Fuzzy Number (Shanmugasundari and Ganesan 2013): A fuzzy set $\tilde{A}$ defined on the set of real numbers R is said to be a fuzzy number if its membership function $\mu_{\tilde{A}} : R \to [0,1]$ has the following characteristics

$\tilde{A}$ is normal. It means that there exists an $x \in R$ such that $\mu_{\tilde{A}}(x) = 1$

(ii) $\tilde{A}$ is convex. It means that for every $x_1, x_2 \in R$, $\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \min.\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\}$, $\lambda \in [0,1]$

(iii) $\mu_{\tilde{A}}$ is upper semi-continuous.

(iv) supp ($\tilde{A}$) is bounded in R

Triangular Fuzzy Number (Gani and Assarudeen 2012):

It is a fuzzy number represented with three points as follows: $\tilde{A} = (a_1, a_2, a_3)$

This representation is interpreted as membership functions and holds the following conditions

(i) a_1 to a_2 is increasing function

(ii) a_2 to a_3 is decreasing function

(iii) $a_1 \leq a_2 \leq a_3$.

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & \text{for } x < a_1 \\ \dfrac{x - a_1}{a_2 - a_1} & \text{for } a_1 \leq x \leq a_2 \\ \dfrac{a_3 - x}{a_3 - a_2} & \text{for } a_2 \leq x \leq a_3 \\ 0 & \text{for } x > a_3 \end{cases}$$

Positive triangular fuzzy number (Gani and Assarudeen 2012)

A positive triangular fuzzy number $\tilde{A}$ is denoted as $\tilde{A} = (a_1, a_2, a_3)$ where all a_i's > 0 for all i=1, 2, 3.

Negative triangular fuzzy number (Gani and Assarudeen 2012)

A negative triangular fuzzy number $\tilde{A}$ is denoted as $\tilde{A} = (a_1, a_2, a_3)$ where all a_i's < 0 for all i=1, 2, 3.

Note: A negative Triangular fuzzy number can be written as the negative multiplication of a positive Triangular fuzzy number. Example: $\tilde{A} = (-3, -2, -1)$ is a negative triangular fuzzy number this can be written as $\tilde{A} = -(1,2,3,)$.

Operation of Triangular Fuzzy Number Using Function Principle (Gani and Assarudeen 2012):

The following are the four operations that can be performed on triangular fuzzy numbers: Let $\tilde{A} = (a_1, a_2, a_3)$ and $\tilde{B} = (b_1, b_2, b_3)$ then,

i) Addition: $\tilde{A} + \tilde{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$.

ii) Subtraction: $\tilde{A} - \tilde{B} = (a_1 - b_3, a_2 - b_2, a_3 - b_1)$.

iii) Multiplication: $\tilde{A} \times \tilde{B} = (\min(a_1 b_1, a_1 b_3, a_3 b_1, a_3 b_3), a_2 b_2, \max(a_1 b_1, a_1 b_3, a_3 b_1, a_3 b_3))$.

iv) Division: $\tilde{A} / \tilde{B} = (\min(a_1/b_1, a_1/b_3, a_3/b_1, a_3/b_3), a_2/b_2, \max(a_1/b_1, a_1/b_3, a_3/b_1, a_3/b_3))$.

3. PROPOSED METHOD

Vogel's Approximation Method for Fuzzy Transportation Problem (Shanmugasundari and Ganesan 2013)

Step 1: From the Fuzzy Transportation table, determine the penalty for each row and column. Identify the boxes having minimum and next to minimum transportation cost in each row & column and write the difference (penalty) along the side of the table against the corresponding row.

Step 2: Identify the largest fuzzy penalty. If it is along the side of the table, make maximum allotment at the box having minimum cost of transportation in that row. If it is below the table make maximum allotment to the box having minimum cost of transportation in that column.

Step 3: If the penalties corresponding to two or more rows or columns are equal, then select any fuzzy penalty.

Modi Method for Fuzzy Transportation Problem (Shanmugasundari and Ganesan 2013)

Step 1: Given an initial fuzzy basic feasible solution of a fuzzy transportation problem in the form of allocated and unallocated cells of fuzzy transportation table.

Step 2: Calculate dual variables $\tilde{p}_i$, i= 1,2,….,m and $\tilde{q}_j$, j= 1,2,….,n for rows and columns respectively by using the following relationship $c_{ij} = \tilde{p}_i + \tilde{q}_j$

Step 3: The fuzzy opportunity cost of an unoccupied cell is determined by using the following relationship $\tilde{d}_{ij} = \tilde{c}_{ij} - (\tilde{p}_i + \tilde{q}_j)$

Step 4:

i. If all $\tilde{d}_{ij} \geq \tilde{0}$, the solution fuzzy optimal.

ii. If any $\tilde{d}_{ij} \prec \tilde{0}$, the solution is not fuzzy optimal and go to step 5.

Step 5: Select most negative $\tilde{d}_{ij}$ among unoccupied cells. Construct closed loop for the unoccupied cell with most negative $\tilde{d}_{ij}$ and mark(+) and (-) sign alternatively beginning with plus sign for the selected unoccupied cell in clockwise or other direction.

Step 6: Assign as many units as possible to the unoccupied cell satisfying rim conditions. The smallest allocation in a cell with negative sign on the closed path indicated the number of units that can be transported to the unoccupied cells. This quantity is added to all the occupied cells on the path marked with plus sign and subtracted from those occupied cells on the path marked with minus signs.

Step 7: Check whether the number of nonnegative allocations is (m+n-1) and repeat step 2 to 6 until $\tilde{d}_{ij} \geq \tilde{0}$.

Operations for Subtraction and Division on Triangular Fuzzy Number (Gani and Assarudeen 2012):

Subtraction:

Let $\tilde{A} = (a_1, a_2, a_3)$ and $\tilde{B} = (b_1, b_2, b_3)$ then, $\tilde{A} - \tilde{B} = (a_1 - b_1, a_2 - b_2, a_3 - b_3)$

The new subtraction operation exist only if the following condition is satisfied $DP(\tilde{A}) \geq DP(\tilde{B})$ where $DP(\tilde{A}) = \frac{a_3 - a_1}{2}$, $DP(\tilde{B}) = \frac{b_3 - b_1}{2}$ and DP denotes Difference point of a Triangular fuzzy number.

Division:

Let $\tilde{A}=(a_1, a_2, a_3)$ and $\tilde{B}=(b_1, b_2, b_3)$ then, $\tilde{A}/\tilde{B}=(a_1/b_1, a_2/b_2, a_3/b_3)$

The new Division operation exists only if the following conditions are satisfied $\left|\frac{DP(\tilde{A})}{MP(\tilde{A})}\right| \geq \left|\frac{DP(\tilde{B})}{MP(\tilde{B})}\right|$ and the negative triangular fuzzy number should be changed into negative multiplication of positive number as per note in definition 2.5 where $MP(\tilde{A})=\frac{a_3+a_1}{2}$, $DP(\tilde{A})=\frac{a_3-a_1}{2}$, $MP(\tilde{B})=\frac{b_3+b_1}{2}$, $DP(\tilde{B})=\frac{b_3-b_1}{2}$ and MP denotes Midpoint of a Triangular fuzzy number.

4. NUMERICAL PROBLEMS

In this section, we will solve two examples 4.1 & 4.2 by using proposed method:

Example 4.1: A company has two factories F_1 & F_2 and two retail stores S_1 & S_2. The production quantities per month at F_1 & F_2 are (150,201,250) & (50,99,150). The demand per month for S_1 & S_2 is (100,149,200) & (100,151,200). The transportation cost per ton c_{11}=(15,24,28), c_{12}=(22,31,35), c_{21}=(8,10,12) & c_{22}=(30,39,54).

Solution: By applying proposed method as discussed in section 3 the Initial fuzzy basic feasible solution is:

Table 1 Initial fuzzy basic feasible solution

	S_1	S_2
F_1	(15,24,28) **(50,50,50)**	(22,31,35) **(100,151,200)**
F_2	(8,10,12) **(50,99,150)**	(30,39,54)

The corresponding initial fuzzy transportation cost is given by IFTC=(15,24,28) x(50,50,50) +(8,10,12)x (50,99,150)+ (22,31,35)x(100,151,200) = (3350,6871,10200). Its defuzzyfied transportation cost is 6839.

By MODI method it can be seen that the current Initial fuzzy basic feasible solution is optimal.

Example 4.2: (Antony et al. 2014)

Consider the following 4X4 Triangular Fuzzy Transportation Problem (TrFTP).

Table 2 Triangular Fuzzy Transportation Problem

	TrFD1	TrFD2	TrFD3	TrFD4	Triangular Fuzzy Supply
TrFO1	(12, 16, 20)	(-1, 1,3)	(6, 8, 10)	(11, 13, 15)	(2, 4, 6)
TrFO2	(7,11,15)	(2,4,6)	(5,7,9)	(6,10,14)	(5,6,7)
TrFO3	(6,8,10)	(12,15,18)	(7,9,11)	(0,2,4)	(7,8,9)
TrFO4	(5,6,7)	(10,12,14)	(3,5,7)	(11,14,17)	(8,10,12)
Triangular Fuzzy Demand	(3,4,5)	(3,5,7)	(10,12,14)	(6,7,8)	

Solution: By applying proposed method as discussed in section 3 the Initial fuzzy basic feasible solution is:

Table 3 Initial fuzzy basic feasible solution

	TrFD1	TrFD2	TrFD3	TrFD4
TrFO1		(-1,1,3)**(2,4,6)**		
TrFO2		(2,4,6) **(1,1,1)**	(5,7,9) **(4,5,6)**	
TrFO3	(6,8,10)**(1,1,1)**			(0,2,4)**(6,7,8)**
TrFO4	(5,6,7) **(2,3,4)**		(3,5,7) **(6,7,8)**	

The corresponding initial fuzzy transportation cost is given by

IFTC=(-1,1,3)(2,4,6) + (2,4,6)(1,1,1) + (5,7,9)(4,5,6) + (6,8,10)(1,1,1) + (0,2,4)(6,7,8) + (5,6,7)(2,3,4) + (3,5,7)(6,7,8) = (50,118,204). Its defuzzyfied transportation cost is 121.

By MODI method it can be seen that the current Initial fuzzy basic feasible solution is optimal.

5. RESULT

In this section, we will compare the transportation cost which has been found out in examples 4.1 & 4.2 in section 4 with the transportation cost solved by existing methods (Shanmugasundari and Ganesan 2013 ; Antony et al. 2014).

Table 4 Comparison of transportation cost

Example 4.1	Existing Method (Shanmugasundari and Ganesan 2013)	6871
	Proposed method	6839
Example 4.2	Existing Method (Antony et al. 2014)	129.67
	Proposed method	121

6. CONCLUSION

In this paper, a new method is proposed by using fuzzy version of VAM and MODI algorithms proposed by Shanmugasundari and Ganesan (2013) and the Operations for Subtraction and Division on Triangular Fuzzy Number proposed by Gani and Assarudin (2012). By comparing the results of the proposed method and existing method, it is shown that it is better to use the proposed methods instead of existing method. Also proposed method is easy to understand and apply for fully fuzzy transportation problem occuring in real life situation.

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MATRIX INVERSION : REPRESENTATION OF NUMERICAL DATA BY A POLYNOMIAL CURVE

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ABSTRACT :
The interpolation by method which consists of the representation of numerical data by a suitable polynomial and then to compute the value of the dependent variable from the polynomial corresponding to any given value of the independent variable leads to the necessity of a method for representing a given set of numerical data on a pair of variables by a suitable polynomial. One such method has been composed in this study. The method is based on the inversion of a square matrix from Caley-Hamilton theorem. The method obtained has been applied in representing the numerical data, on the total population of India since 1971, by a polynomial curve.
Keywords : ***Polynomial curve, representation of numerical data, Caley-Hamilton theorem, interpolation.***

1. INTRODUCTION:

Interpolation is a technique of estimating approximately the value of the dependent variable corresponding to a value of the independent variable lying between its two extreme values on the basis of the given values of the independent and the dependent variables {Hummel (1947), Erdos & Turan (1938) et al}. A number of interpolation formulae are available in the literature of numerical analysis {Bathe & Wilson (1976), Jan (1930), Hummel (1947) et al}.

In case of the interpolation by the existing formulae, if it is wanted to interpolate the values of the dependent variable corresponding to a number of values of the independent variable by a suitable existing interpolation formula then it is required to apply the formula for each value separately and thus the numerical computation of the value of the dependent variable based on the given data are to be performed in each of the cases. In order to get rid of these repeated numerical computations from the given data, one can think of an approach which consists of the representation of the given numerical data by a suitable polynomial and then to compute the value of the dependent variable from the polynomial corresponding to any given value of the independent variable. However, a method is necessary for representing a given set of numerical data *on a pair of variables* by a suitable polynomial. One such method has been composed in this study. The *method is based on the inversion of a square matrix from* Caley-Hamilton theorem *[*Cayley *(1858, 1889) & Hamilton (1864a , 1864b , 1862)]. The method obtained has been applied in representing the numerical data, on the total population of India since 1971, by a polynomial curve.*

2. REPRESENTATION OF NUMERICAL DATA BY POLYNOMIAL CURVE:

Let $y_1, y_2, \ldots\ldots\ldots y_n$ be the values assumed by the function $y = f(x)$ corresponding to the values $x_1, x_2, \ldots\ldots\ldots x_n$ of the independent variable X. Now the problem is to interpolate the values of the function corresponding to some value of x which values of the function is not available. Now the interpolation is based on the mathematical fact that if n points $x_0, x_1, \ldots\ldots, x_n$ are given then they can be suitably represented by a polynomial of degree n-1. In the present case thus the function $y= f(x)$ can be represented suitably by a polynomial in x of degree n. Suppose that the polynomial is $y= f(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_{n-1}x^{n-1}$, $a_{n-1} \neq 0$

Since the n points lie on the polynomial curve describe above so we have

$y_0 = a_0 + a_1 x_0 + a_2 x_0^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_{n-1} x_0^{n-1}$

$y_1 = a_0 + a_1 x_1 + a_2 x_1^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_{n-1} x_1^{n-1}$

$y_2 = a_0 + a_1 x_2 + a_2 x_2^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_{n-1} x_2^{n-1}$

$y_3 = a_0 + a_1 x_3 + a_2 x_3^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_{n-1} x_3^{n-1}$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$$y_n = a_0 + a_1 x_n + a_2 x_n^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_{n-1} x_n^{n-1} \qquad \rightarrow (2.1)$$

In order to obtain the polynomial we are to solve this n equations for the n unknown coefficients (parameters) $a_0, a_1, \ldots\ldots\ldots\ldots, a_n$.

Now solving these equations for the parameters, the polynomial can be obtained.

In order to solve the equations it is to be noted that the equations in (2.1) can be expressed as

$$A\,X = B \qquad \rightarrow (2.2)$$

where

$$A = \begin{pmatrix} 1 & x_0 & x_0^2 \ldots\ldots\ldots\ldots\ldots x_0^{n-1} \\ 1 & x_1 & x_1^2 \ldots\ldots\ldots\ldots\ldots x_1^{n-1} \\ \ldots & \ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots & \ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 1 & x_n & x_n^2 \ldots\ldots\ldots\ldots\ldots x_n^{n-1} \end{pmatrix}, \quad X = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_{n-1} \end{pmatrix} \quad \& \quad B = \begin{pmatrix} y_0 \\ y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{pmatrix} \rightarrow (2.3)$$

which gives $X = A^{-1}B \qquad \rightarrow (2.4)$

Thus in order to find out X*, it is required to find out* A^{-1}.

Now, $|A_n - \lambda I_n|$ *is the characteristic polynomial and*

$$|A_n - \lambda I_n| = 0 \qquad \rightarrow (2.5)$$

is the characteristic equation of the matrix A.

Let the characteristic equation of A be

$$\lambda^n + c_1 \lambda^{n-1} + c_2 \lambda^{n-2} + c_3 \lambda^{n-3} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + c_n = 0 \qquad \rightarrow (2.6)$$

By Caley-Hamilton theorem,

$$A^n + c_1 A^{n-1} + c_2 A^{n-2} + c_3 A^{n-3} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + c_n I_n = 0 \qquad \rightarrow (2.7)$$

Therefore,

$$A^{-1} = -\frac{1}{c_n}[A^{n-1} + c_1 A^{n-2} + c_2 A^{n-3} + c_3 A^{n-4} + \ldots\ldots\ldots + c_{n-1} I_n] \qquad \rightarrow (2.8)$$

Thus in order to find out A^{-1}*, it is required to find out the characteristic polynomial and the characteristic equation of A .*

Now in order to find out the characteristic polynomial of A, let us first consider the matrix

$$A_2 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

of order 2.

The characteristic polynomial of this matrix is given by

$$|A_2 - \lambda I_2| = \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21})$$

$$= (-1)^2 \lambda^2 + (-1)^1 \left(\sum_{i=1}^{2} a_{ii}\right)\lambda + (-1)^0 \left(\sum\sum_{\substack{i=1,j=1 \\ i \neq j}}^{2} a_{ii}a_{jj} - \sum_{i=1}^{2}\sum_{j=1}^{2} a_{ij}a_{ji}\right) \qquad \rightarrow (2.9)$$

The characteristic polynomial of the matrix

$$A_3 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

of order 3 is given by

$$|A_3 - \lambda I_3| = \left| \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right|$$

$$= -\lambda^3 + (a_{11} + a_{22} + a_{33})\lambda^2 - \{(a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33}) - (a_{12}a_{21} + a_{13}a_{31} + a_{23}a_{32})\}\lambda + \{a_{11}a_{22}a_{33} - (a_{11}a_{23}a_{32} + a_{22}a_{13}a_{31} + a_{33}a_{12}a_{21}) + (a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21})\}$$

$$= (-1)^3\lambda^3 + (-1)^2\left(\sum_{i=1}^{3} a_{ii}\right)\lambda^2 + (-1)^1\left\{\sum\sum_{i<j=1}^{3} a_{ii}a_{jj} - \sum\sum_{i\neq j=1}^{3} a_{ij}a_{ji}\right\}\lambda + (-1)^0\left\{\sum\sum\sum_{i<j<k=1}^{3} a_{ij}a_{jj}a_{kk} - \sum_{i=1}^{3} a_{ii}\left(\sum\sum_{j<k=1}^{3} a_{jk}a_{kj}\right) + \sum\sum\sum_{\substack{i\neq j\neq k=1 \\ i\neq j\neq k}}^{3} a_{ij}a_{jk}a_{ki}\right\}$$

$\rightarrow$ *(2.10)*

Similarly the characteristic polynomial of the matrix

$$A_4 = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

of order 4 is given by

$$|A_4 - \lambda I_4| = \left| \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right|$$

$$= (-1)^4\lambda^4 + (-1)^3\left(\sum_{i=1}^{4} a_{ii}\right)\lambda^3 + (-1)^2\left\{\sum\sum_{i<j=1}^{4} a_{ii}a_{jj} - \sum\sum_{\substack{i<j=1 \\ i\neq j}}^{4} a_{ij}a_{ji}\right\}\lambda^2$$

$$+ (-1)^1\left\{\sum\sum\sum_{i<j<k=1}^{4} a_{ij}a_{jj}a_{kk} - \sum_{i=1}^{4} a_{ii}\left(\sum_{\substack{j=1 \\ j\neq k}}^{4}\sum_{k=1}^{4} a_{jk}a_{kj}\right) + \sum_{i=1}^{4}\sum_{j=1}^{4}\sum_{k=1}^{4} a_{ij}a_{jk}a_{ki}\right\}\lambda + \left[\left\{\sum\sum\sum\sum_{i,j,k,l=1}^{4} a_{ii}a_{jj}a_{kk}a_{ll} - \right.\right.$$

$$\sum\sum_{i<j=1}^{4} a_{ii}a_{jj}\left(\sum\sum_{i\neq k=1}^{4} a_{kl}a_{lk}\right) + \sum\sum\sum\sum_{i\neq j\neq k\neq l=1}^{4} a_{ij}a_{jk}a_{kl}a_{li}\Big\} +$$

$$\sum_{i=1}^{4} a_{ii}\left(\sum\sum\sum_{j\neq k\neq l=1}^{4} a_{jk}a_{kl}a_{li}\right) + \sum\sum\sum\sum_{i\neq j\neq k\neq l=1}^{4} a_{ij}a_{kl}a_{lk}a_{ji}\Big\}$$

$\rightarrow$ *(2.11)*

By algebraic continuation, it can be obtained that *the characteristic polynomial of the matrix*

$$A_n = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \ldots\ldots\ldots\ldots\ldots\ldots a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} \ldots\ldots\ldots\ldots\ldots\ldots a_{2n} \\ a_{31} & a_{32} & a_{33} & a_{34} \ldots\ldots\ldots\ldots\ldots\ldots a_{3n} \\ \ldots & \ldots & \ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots & \ldots & \ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} \ldots\ldots\ldots\ldots\ldots\ldots a_{nn} \end{pmatrix}$$

of order n is given by

$$|A_n - \lambda I_n| = (-1)^n\lambda^n + (-1)^{n-1}\left(\sum_{i=1}^{4} a_{ii}\right)\lambda^{n-1} + (-1)^{n-2}\left\{\sum\sum_{i<j=1}^{n} a_{ii}a_{jj} - \right.$$

$$\sum\sum_{i<j=1}^{n} a_{ij}a_{ji} \} \lambda^{n-2} + (-1)^{n-3} \{ \sum\sum\sum_{i<j<k=1}^{n} a_{ij}a_{jj}a_{kk} - \sum_{i=1}^{n} a_{ii} (\sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}a_{kj})$$

$$j\neq k$$

$$+ \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} a_{ij}a_{jk}a_{ki} \} \lambda^{n-3} + (-1)^{n-4} [\{\sum\sum\sum\sum_{i,j,k,l=1}^{n} a_{ii}\, a_{jj}a_{kk}a_{ll} -$$

$$i\neq j\neq k$$

$$\sum\sum_{i<j=1}^{n} a_{ii}a_{jj} (\sum\sum_{i\neq k=1}^{n} a_{kl}\, a_{lk}) + \sum\sum\sum\sum_{i\neq j\neq k\neq l=1}^{n} a_{ij}a_{jk}a_{kl}a_{li} \} +$$

$$\sum_{i=1}^{n} a_{ii} (\sum\sum\sum_{j\neq k\neq l=1}^{n} a_{jk}\, a_{kl}a_{li}) + \sum\sum\sum\sum_{i\neq j\neq k\neq l=1}^{n} a_{ij}a_{kl}a_{lk}a_{ji}] \lambda^{n-4} +$$

$$(-1)^{n-5} [\{ \sum\sum\sum\sum\sum_{i,j,k,l,m=1}^{n} a_{ii}\, a_{jj}a_{kk}a_{ll}a_{mm} -$$

$$\sum\sum\sum_{i<j<k=1}^{n} a_{ii}a_{jj}\, a_{kk} (\sum\sum\sum_{l\neq m\neq k=1}^{n} a_{kl}\, a_{lm}\, a_{mk}) +$$

$$\sum\sum\sum\sum\sum_{i\neq j\neq k\neq l\neq m=1}^{n} a_{ij}a_{jk}a_{kl}a_{lm}\, a_{mi} \} +$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ii}\; a_{jj} (\sum\sum\sum\sum_{j\neq k\neq l\neq m=1}^{n} a_{jk}\; a_{kl}a_{lm}a_{mj}) +$$

$$\sum\sum\sum\sum\sum_{i\neq j\neq k\neq l\neq m=1}^{n} a_{ij}a_{kl}a_{lk}a_{km}\, a_{mi}] \lambda^{n-5} + \ldots\ldots\ldots\ldots$$

$$\rightarrow (2.12)$$

Therefore, the characteristics polynomial of the matrix A becomes

$$|A-\lambda I| = = (-1)^{n} \lambda^{n} + (-1)^{n-1} (\sum_{i=1}^{4} 1) \lambda^{n-1} + (-1)^{n-2} \{ \sum\sum_{i<j=1}^{n} 1\,.\,x_1 -$$

$$\sum\sum_{i<j=1}^{n} x_{i-1}^{j-1}\, x_{j-1}^{i-1} \} \lambda^{n-2} + (-1)^{n-3} \{ \sum\sum\sum_{i<j<k=1}^{n} x_{i-1}^{j-1}.x_1x_2^2 - \sum_{i=1}^{n} 1 (\sum_{j=1}^{n}\sum_{k=1}^{n} x_{j-1}^{k-1}x_{k-1}^{j-1})$$

$$j\neq k$$

$$+ \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} x_{i-1}^{j-1}x_{j-1}^{k-1}x_{k-1}^{i-1} \} \lambda^{n-3} + (-1)^{n-4} [\{\sum\sum\sum\sum_{i,j,k,l=1}^{n} 1\, x_1x_2^2x_2^3 -$$

$$i\neq j\neq k$$

$$\sum\sum_{i<j=1}^{n} 1.\,x_1 (\sum\sum_{i\neq k=1}^{n} x_{k-1}^{l-1}\, x_{l-1}^{k-1}) + \sum\sum\sum\sum_{i\neq j\neq k\neq l=1}^{n} x_{i-1}^{j-1}x_{j-1}^{k-1}x_{k-1}^{l-1}x_{l-1}^{i-1} \} +$$

$$\sum_{i=1}^{n} 1 (\sum\sum\sum_{j\neq k\neq l=1}^{n} x_{j-1}^{k-1}\, x_{k-1}^{l-1}x_{l-1}^{i-1}) + \sum\sum\sum\sum_{i\neq j\neq k\neq l=1}^{n} x_{i-1}^{j-1}x_{k-1}^{l-1}x_{l-1}^{k-1}x_{j-1}^{i-1}] \lambda^{n-4} +$$

$$(-1)^{n-5} [\{ \sum\sum\sum\sum\sum_{i,j,k,l,m=1}^{n} 1\;\; x_1x_2^2x_3^3x_4^4 -$$

$$\sum\sum\sum_{i<j<k=1}^{n} 1.\,x_1\, x_2^2 (\sum\sum\sum_{l\neq m\neq k=1}^{n} x_{k-1}^{l-1}\, x_{l-1}^{m-1}\, x_{m-1}^{k-1}) +$$

$$\sum\sum\sum\sum\sum_{i\neq j\neq k\neq l\neq m=1}^{n} x_{i-1}^{j-1}x_{j-1}^{k-1}x_{k-1}^{l-1}x_{l-1}^{m-1}\, x_{m-1}^{i-1} \} +$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} 1\;\, .\,x_1 (\sum\sum\sum\sum_{j\neq k\neq l\neq m=1}^{n} x_{j-1}^{k-1}\;\, x_{k-1}^{l-1}x_{l-1}^{m-1}x_{m-1}^{j-1}) +$$

$$\sum\sum\sum\sum\sum_{i\neq j\neq k\neq l\neq m=1}^{n} x_{i-1}^{j-1} x_{k-1}^{l-1} x_{l-1}^{k-1} x_{k-1}^{m-1} x_{m-1}^{i-1} \,]\, \lambda^{n-5} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$\rightarrow$ *(2.13)*

3. AN EXAMPLE OF APPLICATION OF THE FORMULA:

The following table shows the data on total population of India corresponding to the years:

Year	1971	1981	1991	2001
Total Population	548159652	683329097	846302688	1027015247

Taking 1971 as origin and changing scale by 1/10, one can obtain the following table for independent variable x (representing time) and $f(x)$ (representing total population of India):

Year	1971	1981	1991	2001
x_i	0	1	2	3
$f(x_i)$	548159652	683329097	846302688	1027015247

Now here $x_0 = 0,\ x_1 = 1,\ x_2 = 2,\ x_3 = 3$

$f(x_0) = 548159652,\ f(x_1) = 683329097,\ f(x_2) = 846302688,\ f(x_3) = 1027015247,$

$$\text{Here } X = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{pmatrix}, \quad B = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$|A - \lambda I| = \left| \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right|$$

$$= \left| \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix} \right|$$

$$= \begin{vmatrix} 1-\lambda & 0 & 0 & 0 \\ 1 & 1-\lambda & 1 & 1 \\ 1 & 2 & 4-\lambda & 8 \\ 1 & 3 & 9 & 27-\lambda \end{vmatrix}$$

$$= (1-\lambda) \begin{vmatrix} 1-\lambda & 1 & 1 \\ 2 & 4-\lambda & 8 \\ 3 & 9 & 27-\lambda \end{vmatrix}$$

$$= (1-\lambda)\,[(1-\lambda)\{(4-\lambda)(27-\lambda) - 72\} - 1\{2(27-\lambda) - 24\} + 1\{18 - 3(4-\lambda)\}]$$
$$= (1-\lambda)\,[(1-\lambda)\,\{108 - 4\lambda - 27\lambda + \lambda^2 - 72\} - \{54 - 2\lambda - 24\} + \{18 - 12 + 3\lambda\}]$$
$$= (1-\lambda)\,[(1-\lambda)\,\{108 - 31\lambda + \lambda^2 - 72\} - \{30 - 2\lambda\} + \{6 + 3\lambda\}]$$
$$= (1-\lambda)[\,(108 - 31\lambda + \lambda^2 - 72) - \lambda\,(108 - 31\lambda + \lambda^2 - 72) - 30 + 2\lambda + 6 + 3\lambda]$$
$$= (1-\lambda)\,[36 - 31\lambda + \lambda^2 - 108\lambda + 31\lambda^2 - \lambda^3 + 72\lambda - 30 + 2\lambda + 6 + 3\lambda]$$
$$= (1-\lambda)\,[-\lambda^3 + 32\lambda^2 - 139\lambda + 77\lambda + 12]$$

$= (1-\lambda)\,[-\lambda^3 + 32\lambda^2 - 62\lambda + 12]$

$= (-\lambda^3 + 32\lambda^2 - 62\lambda + 12) - \lambda(-\lambda^3 + 32\lambda^2 - 62\lambda + 12)$

$= -\lambda^3 + 32\lambda^2 - 62\lambda + 12 + \lambda^4 - 32\lambda^3 + 62\lambda^2 - 12\lambda$

$= \lambda^4 - 33\lambda^3 + 94\lambda^2 - 74\lambda + 12$

$\therefore$ The characteristic equation is

$$\lambda^4 - 33\lambda^3 + 94\lambda^2 - 74\lambda + 12 = 0$$

By Caley Hamilton theorem, we have

$$A^4 - 33A^3 + 94A^2 - 74A + 12I = 0$$

$$\Rightarrow 12I = -A^4 + 33A^3 - 94A^2 + 74A$$

$$\Rightarrow I = \frac{1}{12}[-A^4 + 33A^3 - 94A^2 + 74A]$$

$$\Rightarrow A^{-1} = \frac{1}{12}[-A^3 + 33A^2 - 94A + 74I] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3.1)$$

Now, $A^2 = A.A$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{pmatrix}$$

$$= \begin{pmatrix} 1+0+0+0 & 0+0+0+0 & 0+0+0+0 & 0+0+0+0 \\ 1+1+1+1 & 0+1+2+3 & 0+1+4+9 & 0+1+8+27 \\ 1+2+4+8 & 0+2+8+24 & 0+2+16+72 & 0+2+32+216 \\ 1+3+9+27 & 0+3+18+81 & 0+3+36+243 & 0+3+72+729 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 6 & 14 & 36 \\ 15 & 34 & 90 & 250 \\ 40 & 102 & 282 & 804 \end{pmatrix}$$

$A^3 = A^2.A$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 6 & 14 & 36 \\ 15 & 34 & 90 & 250 \\ 40 & 102 & 282 & 804 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{pmatrix}$$

$$= \begin{pmatrix} 1\times1+0\times1+0\times1+0\times1 & 1\times0+0\times1+0\times2+0\times3 \\ 4\times1+6\times1+14\times1+36\times1 & 4\times0+6\times1+14\times2+36\times3 \\ 15\times1+34\times1+90\times1+250\times1 & 15\times0+34\times1+90\times2+250\times3 \\ 40\times1+102\times1+282\times1+804\times1 & 40\times0+102\times1+282\times2+804\times3 \\ \\ 1\times0+0\times1+0\times4+0\times9 & 1\times0+0\times1+0\times8+0\times27 \\ 4\times0+6\times1+14\times4+36\times9 & 4\times0+6\times1+14\times8+36\times27 \\ 15\times0+34\times1+90\times4+250\times9 & 15\times0+34\times1+90\times8+250\times27 \\ 40\times0+102\times1+282\times4+804\times9 & 40\times0+102\times1+282\times8+804\times27 \end{pmatrix}$$

$$= \begin{pmatrix} 1+0+0+0 & 0+0+0+0 & 0+0+0+0 & 0+0+0+0 \\ 4+6+14+36 & 0+6+28+108 & 0+6+56+324 & 0+6+112+972 \\ 15+34+90+250 & 0+34+180+750 & 0+34+360+2250 & 0+34+720+6750 \\ 40+102+282+804 & 0+102+564+2412 & 0+102+1128+7236 & 0+102+2256+21708 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 60 & 142 & 386 & 1090 \\ 389 & 964 & 2644 & 7504 \\ 1228 & 3078 & 8466 & 24066 \end{pmatrix}$$

$$\therefore (1) \Rightarrow A^{-1} = \frac{1}{12}\left[-A^3 + 33\,A^2 - 94\,A + 74\,I \right]$$

$$= \frac{1}{12}\left[-\begin{pmatrix} 1 & 0 & 0 & 0 \\ 60 & 142 & 386 & 1090 \\ 389 & 964 & 2644 & 7504 \\ 1228 & 3078 & 8466 & 24066 \end{pmatrix} + 33\begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 6 & 14 & 36 \\ 15 & 34 & 90 & 250 \\ 40 & 102 & 282 & 804 \end{pmatrix} \right.$$

$$\left. - 94\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{pmatrix} + 74\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right]$$

$$= \frac{1}{12}\left[-\begin{pmatrix} 1 & 0 & 0 & 0 \\ 60 & 142 & 386 & 1090 \\ 389 & 964 & 2644 & 7504 \\ 1228 & 3078 & 8466 & 24066 \end{pmatrix} + \right.$$

$$\begin{pmatrix} 33 & 0 & 0 & 0 \\ 132 & 198 & 462 & 1188 \\ 495 & 1122 & 2970 & 8250 \\ 1320 & 3366 & 9306 & 26532 \end{pmatrix}$$

$$\left. - \begin{pmatrix} 94 & 0 & 0 & 0 \\ 94 & 94 & 94 & 94 \\ 94 & 188 & 376 & 752 \\ 94 & 282 & 846 & 2538 \end{pmatrix} + \begin{pmatrix} 74 & 0 & 0 & 0 \\ 0 & 74 & 0 & 0 \\ 0 & 0 & 74 & 0 \\ 0 & 0 & 0 & 74 \end{pmatrix} \right]$$

$$= \frac{1}{12}\begin{pmatrix} -1+33-94+74 & 0+0+0+0 & 0+0-0+0 & 0+0+0+0 \\ -60+132-94+0 & -142+198-94+74 & -386+462-94+0 & -1090+1188-94+0 \\ -389+495-94+0 & -964+1122-188+0 & -2644+2970-376+74 & -7504+8250-752+0 \\ -1228+1320-94+0 & -3078+3366-282+0 & -8466+9306-846+0 & -24066+26532-2538+74 \end{pmatrix}$$

$$= \frac{1}{12}\begin{pmatrix} 12 & 0 & 0 & 0 \\ -22 & 36 & -18 & 4 \\ 12 & -30 & 24 & -6 \\ -2 & 6 & -6 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{12}{12} & 0 & 0 & 0 \\ \frac{-22}{12} & \frac{36}{12} & \frac{-18}{12} & \frac{4}{12} \\ \frac{12}{12} & \frac{-30}{12} & \frac{24}{12} & \frac{-6}{12} \\ \frac{-2}{12} & \frac{6}{12} & \frac{-6}{12} & \frac{2}{12} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{-11}{6} & 3 & \frac{-3}{2} & \frac{1}{-3} \\ 1 & \frac{-5}{2} & 2 & \frac{-1}{2} \\ \frac{-1}{6} & \frac{1}{2} & \frac{-1}{2} & \frac{1}{6} \end{pmatrix}$$

$$\therefore X = A^{-1}B$$

$$\Rightarrow \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{-11}{6} & 3 & \frac{-3}{2} & \frac{1}{3} \\ 1 & \frac{-5}{2} & 2 & \frac{-1}{2} \\ \frac{-1}{6} & \frac{1}{2} & \frac{-1}{2} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} 548159652 \\ 683329097 \\ 846302688 \\ 1027015247 \end{pmatrix}$$

$$= \begin{pmatrix} 548159652 \ + 0 + 0 + 0 \\ \frac{-11}{6} \times 548159652 + 3 \ \times \ 683329097 + \frac{-3}{2} \times \ 846302688 + \frac{1}{3} \times 1027015247 \\ 1 \times 548159652 + \frac{-5}{2} \times 683329097 \ + 2 \times 846302688 + \quad \frac{-1}{2} \times \ 1027015247 \\ \frac{-1}{6} \times 548159652 + \frac{1}{2} \times 683329097 + \frac{-1}{2} \times 846302688 + \frac{1}{6} \times 1027015247 \end{pmatrix}$$

$$= \begin{pmatrix} 548159652 \\ -1004959362 + 2049987291 - 1269454032 + 342338415.66 \\ 548159652 - 1708322742.5 + \ 1692605376 - 513507623.5 \\ -91359942 + \ 341664548.5 - \ 423151344 + \ 171169207.83 \end{pmatrix}$$

$$= \begin{pmatrix} 548159652 \\ 117912312.66 \\ 18934662 \\ -1677529.67 \end{pmatrix}$$

$\therefore$ a_0 = 548159652 , a_1 = 117912312.66, a_2 = 18934662, a_3= −1677529.67

$\therefore$ The Polynomial equations are

$y_0 = f(x_0) = \ a_0 + a_1 x_0 + a_2 \ {x_0}^2 \ + a_3 \ {x_0}^3$

$y_1 = f(x_1) = a_0 + a_1 x_1 + a_2 \ {x_1}^2 \ + a_3 \ {x_1}^3$

$y_2 = f(x_2) = a_0 + a_1 x_2 + a_2 \ {x_2}^2 \ + a_3 \ {x_2}^3$

$y_3 = f(x_3) = a_0 + a_1 x_3 + a_2 \ {x_3}^2 \ + a_3 \ {x_3}^3$

Now, putting the values of a_0, a_1, a_2, a_3 in the polynomial equations, we have

$f(x_0)$ = 548159652

$f(x_1)$ = 548159652 +117912312.66 + 18934662 −1677529.67

= 685006626.66 −1677529.67

= 683329096.99

= 683329097

$f(x_2)$ = 548159652 +117912312.66 × 2 + 18934662 × 4 − 1677529.67 × 8

= 548159652 +235824625.32 + 75738648 − 13420237.36

= 859722925.32 − 13420237.36

= 846302687.96

= 846302688

$f(x_3)$ = 548159652 +117912312.66 × 3 + 18934662 × 9 − 1677529.67 × 27

= 548159652 +353736937.98 + 170411958 − 45293301.09

= 1072308547.98 − 45293301.09

= 1027015246.89

= 1027015247

5. CONCLUSION:

The method composed here can be suitably used to represent a given set of numerical data on a pair of variables by a polynomial. The degree of the polynomial is one less than the number of pairs of observations.

The polynomial that represents the given set of numerical data can be used for interpolation at any position of the independent variable lying within its two extreme values.

The approach of interpolation, described here, can be suitably applied in inverse interpolation also.

It has already been mentioned that in case of the interpolation by the existing formulae, if it is wanted to interpolate the values of the dependent variable corresponding to a number of values of the independent variable by a suitable existing interpolation formula then it is required to apply the formula for each value separately and thus the numerical computation of the value of the dependent variable based on the given data are to be performed in each of the cases. The method developed here has rescued from these repeated numerical computations from the given data.

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STOCHASTIC ANALYSIS OF A SYSTEM WITH PATIENCE TIME AND TELEPHONIC DISCUSSION REGARDING NATURE OF FAILURE TO DECIDE TYPE OF VISITING REPAIRMAN

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ABSTRACT :
A reliability model of a two unit cold standby system is discussed wherein a regular repairman remains always with the system and may get tired while repairing a failed unit. If the regular repairman gets tired, expert repairman first discusses the nature of failure telephonically to decide whether he himself should go or to send his assistant repairman. If the assistant / regular repairman is unable to repair the failed unit, then the expert takes his own time to arrive at the system. However, on the completion of patience time or when the system becomes inoperable, the expert comes immediately to undertake the system and repairs all the units which fail during his stay at the system.

Keywords: *cold standby, regular repairman, expert repairman, assistant repairman, patience time*

I. INTRODUCTION

Literature in the field of reliability is rich with various models on two-unit standby systems discussed under numerous situations. A good number of researchers including [1-5] discussed models taking rest period of repairman. Taneja and Gupta [6] analysed a system with rest period of regular repairman and probability of type of visiting repairman , an expert or his assistant, who comes to repair the system when regular repairman gets tired or declares himself unable to complete the repair. They also considered that:

i) The assistant/regular repairman is allowed to take his own time to repair till he declares himself unable to complete the repair;
ii) The expert repairman comes immediately as and when the regular/assistant repairman declares himself unable to complete the repair or otherwise required.

However, waiting for the expert till the assistant declares himself unable to repair may cause to the great losses to the system due to its long unavailability and hence there is need to limit the time (called patience time) during which the assistant tries to complete the repair. A good number of studies taking rest/down or need based operation period and/or various types of repairman were made by a large number of researchers including [7-10], but none of these studies include the aspect of patience time while the assistant tries to complete the repair.

Incorporating the idea of such patience time together with the concepts taken in the paper [6], a two-unit cold standby system has been analyzed. In which a regular repairman always remains at the system and may get tired or may not be able to repair the failed unit within patience time. When the regular repairman get tired, the expert repairman discusses about the nature of failure telephonically to decide whether he himself should go to the system or send his assistant to repair the failed unit. If regular / assistant repairman is not able to repair the failed unit, then expert himself comes to resume the system.

Furthermore, the expert repairman comes to the system immediately on the expiry of the patience time or whenever the system becomes inoperable and repairs all the units which fails during his stay at the system.

The model discussed in the present paper is also compared with that discussed in [6].

II. MATERIALS AND METHODS

Various measures of system effectiveness such as Mean Time to System Failure (MTSF), steady state availability, total fraction of the time for which the assistant/expert repairman is busy, expected number of visits by the assistant/expert repairman, expected discussion time are found out. Using the above measures, expression for the profit is obtained and graphical study is made for a particular case.

III. NOTATIONS

λ	constant failure rate
a	probability that the regular/assistant repairman is able to complete the repair
b	probability that the regular repairman does not need rest
p	= ab
q	= a(1−b)
p_1	probability that the regular repairman is available with the system after recovery from rest
q_1	probability that the regular repairman is not available as he is taking rest
p_2	probability that after discussing the nature of failure, expert himself comes to the system
q_2	probability that after discussing the nature of failure, the expert sends his assistant to repair the failed unit.
$g(t), G(t)$	p.d.f. and c.d.f. of the repair time of the regular repairman
$g_a(t), G_a(t)$	p.d.f. and c.d.f. of the repair time of the assistant repairman
$g_e(t), G_e(t)$	p.d.f. and c.d.f. of the repair time of the expert repairman
$i(t), I(t)$	p.d.f. and c.d.f. of the patience time
$w(t), W(t)$	p.d.f. and c.d.f. of waiting time for repair to be done by the expert repairman
$h_2(t), H_2(t)$	p.d.f. and c.d.f. of discussion time
$E_1(t)$	$= g(t)\ \bar{I}(t)$
$E_2(t)$	$= i(t)\ \bar{G}(t)$
$E_3(t)$	$= \bar{I}(t)\ \bar{G}(t)$
$F_1(t)$	$= g_a(t)\ \bar{I}(t)$
$F_2(t)$	$= i(t)\ \bar{G}_a(t)$
$F_3(t)$	$= \bar{I}(t)\ \bar{G}_a(t)$

Symbols for the states of the system are:

o	operative unit
cs	cold standby
o_n	operative unit (suffix ‘n’ represents that regular repairman is not available)
Fr	failed unit under repair of the regular repairman
Fra	failed unit under repair of the assistant repairman
Fre	failed unit under repair of the expert repairman
FRe	repair of the failed unit is continuing by the expert repairman from the previous state
F_w	failed unit waiting for repair to be done by the expert repairman
F_{wd}	failed unit waiting for repair while discussion for knowing the nature of failure is going on

IV. TRANSITION PROBABILITIES AND MEAN SOJOURN TIMES

The transition diagram showing the various states of the system is shown as in Fig. 6.1. The epochs of entry into states 0, 1, 2, 3, 4, 5, 6 and 8 are regeneration points and thus these states are regenerative states. State 5 and 7 are down states. The transition probabilities are:

$q_{01}(t) = \lambda e^{-\lambda t}$ $\qquad$ $q_{10}(t) = pe^{-\lambda t}\, g(t)\ \overline{I}(t) = pe^{-\lambda t}\, E_1(t)$

$q_{12}(t) = qe^{-\lambda t}\, g(t)\ \overline{I}(t) = qe^{-\lambda t}\, E_1(t)$ $\qquad$ $q_{13}(t) = e^{-\lambda t}\, i(t)\ \overline{G}(t) = e^{-\lambda t}\, E_2(t)$

$q_{14}(t) = (1-a)\, e^{-\lambda t}\, g(t)\ \ \overline{I}(t) = (1-a)\, e^{-\lambda t}\, E_1(t)$ $\qquad$ $q_{15}(t) = \lambda e^{-\lambda t}\ \ \overline{I}(t)\ \overline{G}(t) = \lambda e^{-\lambda t}\, E_3(t)$

$q_{24}(t) = p_2 e^{-\lambda t}\, h_2(t)$ $\qquad$ $q_{25}(t) = \lambda e^{-\lambda t}\ \ \overline{H}_2(t)$

$q_{26}(t) = q_2 e^{-\lambda t}\, h_2(t)$ $\qquad$ $q_{30}(t) = p_1 e^{-\lambda t}\, g_e(t)$

$q_{38}(t) = q_1 e^{-\lambda t}\, g_e(t)$ $\qquad$ $q_{37}(t) = \lambda e^{-\lambda t}\ \overline{G}_e(t)$

$q_{33}^{(7)}(t) = [\lambda e^{-\lambda t}$ ©$1]\, g_e(t) = [1 - e^{-\lambda t}]\, g_e(t)$ $\qquad$ $q_{43}(t) = e^{-\lambda t}\, w(t)$

$q_{45}(t) = \lambda e^{-\lambda t}\ \ \overline{W}(t)$ $\qquad$ $q_{53}(t) = g_e(t)$

$q_{60}(t) = p_1\, ae^{-\lambda t}\, g_a(t)\ \overline{I}(t) = p_1\, ae^{-\lambda t}\, F_1(t)$ $\qquad$ $q_{63}(t) = e^{-\lambda t}\, i(t)\ \overline{G}_a(t) = e^{-\lambda t}\, F_2(t)$

$q_{64}(t) = (1-a)\, e^{-\lambda t}\, g_a(t)\ \overline{I}(t) = (1-a)\, e^{-\lambda t}\, F_1(t)$ $\qquad$ $q_{65}(t) = \lambda e^{-\lambda t}\ \overline{G}_a(t)\ \overline{I}(t) = \lambda\, e^{-\lambda t}\, F_3(t)$

$q_{68}(t) = q_1\, ae^{-\lambda t}\, g_a(t)\ \overline{I}(t) = q_1 ae^{-\lambda t}\, F_1(t)$ $\qquad$ $q_{81}(t) = p_1\, \lambda e^{-\lambda t}$

$q_{82}(t) = q_1 \lambda\, e^{-\lambda t}$ $\qquad$ (1-23)

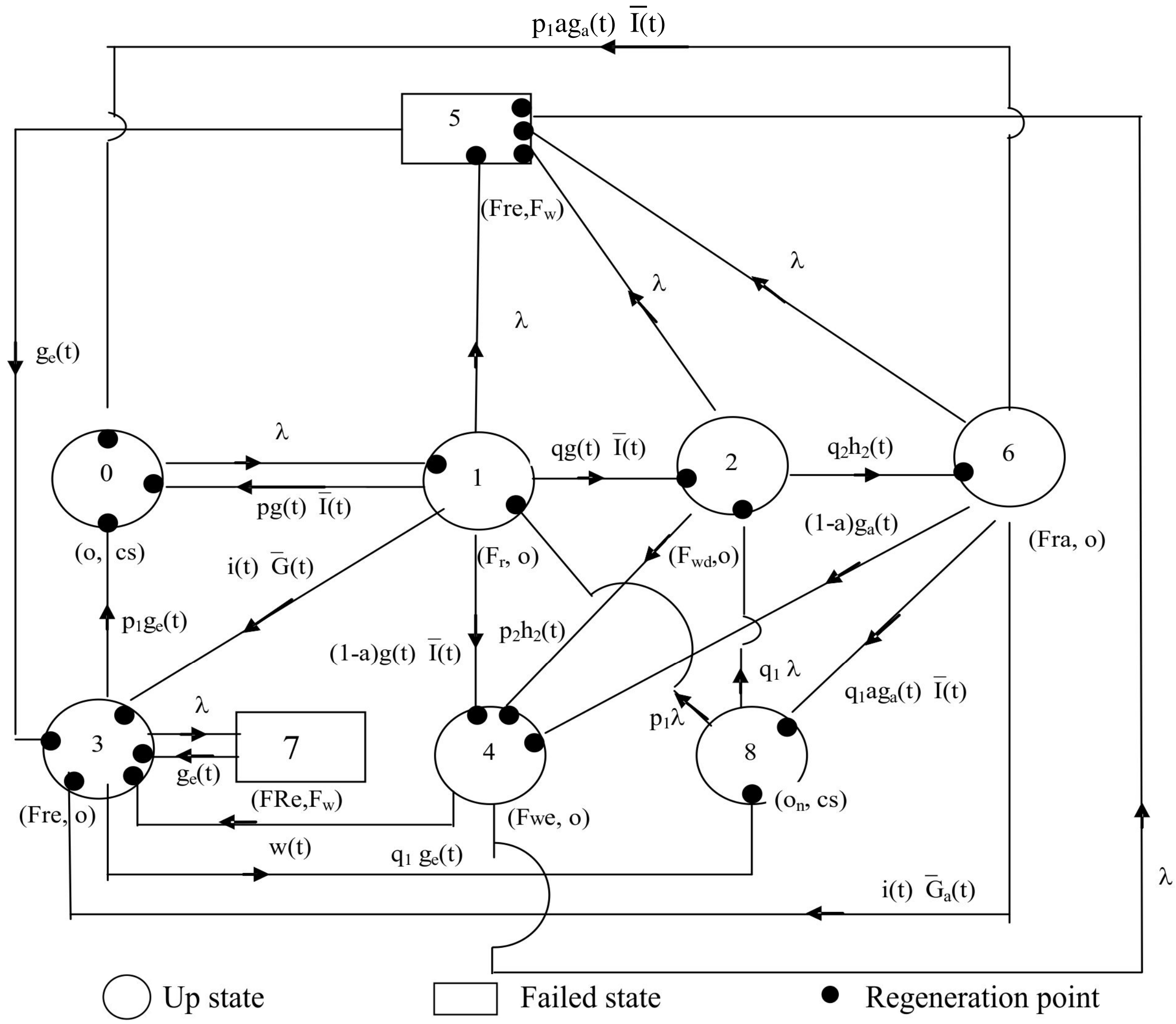

Fig. 6.1 State Transition Diagram

The non-zero elements p_{ij} are given as follows:

$p_{01} = 1$, $p_{10} = p\,E_1^*(\lambda)$, $p_{12} = q\,E_1^*(\lambda)$,

$p_{13} = E_2^*(\lambda)$, $p_{14}\ (1-a)E_1^*(\lambda)$, $p_{15} = \lambda E_3^*(\lambda)$,

$p_{24} = p_2h_2^*(\lambda)$, $p_{25} = 1-h_2^*(\lambda)$, $p_{26} = q_2h_2^*(\lambda)$,

$p_{30} = p_1g_e^*(\lambda)$, $p_{38} = q_1g_e^*(\lambda)$, $p_{37} = 1-g_e^*(\lambda)$,

$p_{33}^{(7)} = 1-g_e^*(\lambda)$, $p_{43} = w^*(\lambda)$, $p_{45} = 1 - w^*(\lambda)$,

$p_{53} = 1$, $p_{60} = p_1a\,F_1^*(\lambda)$, $p_{63} = F_2^*(\lambda)$,

$p_{64} = (1-a)F_1^*(\lambda)$, $p_{65} = \lambda\,F_3^*(\lambda)$, $p_{68} = q_1a\,F_1^*(\lambda)$,

$p_{81} = p_1$, $p_{82} = q_1$ (24-46)

By these transition probabilities, it can be verified that

$p_{01} = 1$, $p_{10} + p_{12} + p_{13} + p_{14} + p_{15} = 1$

$p_{24} + p_{25} + p_{26} = 1$, $p_{30} + p_{37} + p_{38} = 1$

$p_{30} + p_{38} + p_{33}^{(7)} = 1$, $p_{43} + p_{45} = 1$,

$p_{53} = 1$ $p_{60} + p_{63} + p_{64} + p_{65} + p_{68} = 1$,

$p_{81} + p_{82} = 1$ (47-55)

The mean sojourn times μ_i are:

$\mu_0 = \dfrac{1}{\lambda}$, $\mu_1 = E_3^*(\lambda)$, $\mu_2 = \dfrac{(1-h_2^*(\lambda))}{\lambda}$

$\mu_3 = \dfrac{(1-g_e^*(\lambda))}{\lambda}$ $\mu_4 = \dfrac{(1-w^*(\lambda))}{\lambda}$, $\mu_5 = \int_0^\infty t g_e(t)dt = -g_e^{*\prime}(0)$

$\mu_6 = F_3^*(\lambda)$, $\mu_8 = \dfrac{1}{\lambda}$ (56-63)

The unconditional mean time taken by the system to transit for any state j when it is counted from epoch of entrance into state i is mathematically stated as :

$$m_{ij} = \int_0^\infty t\, q_{ij}(t)\, dt = -q_{ij}^{*\prime}(0) \tag{64}$$

Thus,

$m_{01} = \mu_0$, $m_{10} + m_{12} + m_{13} + m_{14} + m_{15} = \mu_1$

$m_{24} + m_{25} + m_{26} = \mu_2$, $m_{30} + m_{37} + m_{38} = \mu_3$

$m_{30} + m_{33}^{(7)} + m_{38} = \mu_5$, $m_{43} + m_{45} = \mu_4$

$m_{53} = \mu_5$, $m_{60} + m_{63} + m_{64} + m_{65} + m_{68} = \mu_6$

$m_{81} + m_{82} = \mu_8$ (65-73)

V. MEAN TIME TO SYSTEM FAILURE

The mean to system failure (MTSF), when the system starts from the starts '0' is

$$T_0 = \lim_{s\to 0}\frac{1-\phi_0^{**}(s)}{s} = \frac{N}{D} \tag{74}$$

where $N = (p_{13} + p_{14}p_{43})\ [\,p_{38}(p_{82}\mu_2 - p_{81}\mu_0) + (1-p_{26}p_{68}p_{82})\mu_3 + p_{38}p_{82}p_{26}\mu_0] + (p_{24} + p_{26}p_{64})\ [(p_{12} + p_{13}p_{38}p_{82})\mu_4 + p_{12}p_{43}\mu_3 - p_{43}p_{38}p_{82}\{\mu_1 + (1-p_{12})\mu_0\}] + p_{12}p_{26}(\mu_6 + p_{63}\mu_3 - p_{68}\mu_0) + (1 + p_{12}p_{26}p_{65})\mu_0 - p_{82}p_{26}(p_{68} + p_{63}p_{38})\{(1-p_{12})\mu_0 + \mu_1 + p_{14}\mu_4\} + \mu_1 + p_{12}\mu_2 + p_{14}\mu_4$

$D = 1 - p_{10} - p_{43}(p_{24} + p_{26}p_{64})\ \{(p_{13} + p_{14} + p_{15})p_{38}p_{82} + (p_{30} + p_{38})\}$
$\quad - p_{26}\{p_{12}(p_{60} + p_{68} + p_{63}p_{30}) + p_{63}p_{38}p_{81}p_{12} + p_{82}(1-p_{10})(p_{68} + p_{63}p_{38})\}$

(75-76)

VI. AVAILABILITY ANALYSIS:

In steady state, the availability of the system is given by

$$A_0 = N_1 / D_1 \tag{77}$$

Where $N_1 = (p_{30} + p_{38})\ [\mu_0\{1-p_{26}p_{68}p_{82}(1-p_{12})\} + (\mu_1 + p_{14}\mu_4)\ (1-p_{26}p_{68}p_{82}) + p_{12}\{(\mu_2 + p_{24}\mu_4) - p_{26}(\mu_6 + p_{64}\mu_4\}] - p_{26}\ [p_{38}\mu_0p_{82}(1-p_{12}) - p_{12}\mu_3]\ [p_{63} + p_{64} + p_{65}] - p_{26}p_{82}p_{38}[p_{63}(\mu_1 + p_{14}\mu_4) + p_{64}\,(\mu_1p_{43} - \mu_4p_{13}) + \{\mu_1(p_{65} + p_{64}p_{45}) + \mu_4(p_{65}p_{14} - p_{15}p_{64})\}] - (p_{82}p_{26} + p_{81}p_{12})p_{38}\,\mu_0(p_{24} + p_{25}) - [p_{38}p_{82}\{\mu_4(p_{14}p_{25} - p_{24}p_{15}) + (p_{25} + p_{24}p_{45})\mu_1\} - p_{12}(p_{25} + p_{24}p_{45})\ (\mu_3 + p_{38}\mu_0) + p_{24}[p_{12}p_{43}(\mu_3 + p_{38}\mu_0) + p_{38}p_{82}(p_{13}\mu_4 - p_{43}\mu_1) + [p_{13} + p_{14} + p_{15}]\ [p_{38}p_{82}(\mu_0 + \mu_2 + p_{26}\mu_6) + (1-p_{26}p_{68}p_{82})\mu_3]$

$D_1 = [\{p_{10}(1-p_{26}p_{68}p_{82}) + p_{12}p_{26}(p_{60} + p_{68})\}\ (p_{30} + p_{38}) + p_{38}(p_{13} + p_{14} + p_{15})\ \{1 + p_{82}p_{26}p_{60}\} + (p_{30}p_{12} - p_{10}p_{38}p_{82})\ \{1-p_{26}(p_{60} + p_{68})\}]\mu_0 + (p_{30} + p_{38})\ [p_{26}(p_{12}\mu_6 + p_{60}\mu_1) + p_{12}\mu_2] + p_{12}p_{26}(p_{63} + p_{64} + p_{65})(\mu_5 + p_{38}\mu_0) + [p_{30}(1-p_{26}p_{68}p_{82}) + p_{38}(p_{81} + p_{82}p_{26}p_{60})]\ [\mu_1 + p_{14}\mu_4 + (p_{15} + p_{14}p_{45})\mu_5] + [p_{38}\{p_{12} + p_{82}(p_{13} + p_{14} + p_{15})\} + p_{30}p_{12}][\mu_4(p_{24} + p_{26}p_{64}) + \mu_5\ \{p_{25} + p_{24}p_{45} + p_{26}(p_{65} + p_{64}p_{45})\}] + p_{12}(p_{24} + p_{45})(\mu_0 + \mu_5) + (p_{13} + p_{14} + p_{15})\ [p_{38}p_{82}(\mu_2 + p_{26}\mu_6) + (1-p_{26}p_{68}p_{82})\mu_5]$ (78-79)

VII. BUSY PERIOD ANALYSIS OF THE EXPERT REPAIRMAN

In steady state, the total fraction of time for which the system is under repair of the expert repairman is given by

$$B_0^e = N_2 / D_1 \tag{80}$$

Where $N_2 = \mu_5[(1-p_{26}p_{68}p_{82})\ (p_{13} + p_{14} + p_{15}) + p_{12}\ (p_{24} + p_{26}p_{64}) + (p_{14}p_{45} + p_{15})\ \{(p_{30} + p_{38})\ (1 + p_{82}p_{26}p_{68}) - p_{63}p_{38}p_{82}p_{26}\} + \{p_{24}p_{45} + p_{25} + p_{26}(p_{65} + p_{64}p_{45})\}\{p_{12}(p_{30} + p_{38}) + p_{13}p_{38}p_{82}\} + p_{12}\{p_{25} + p_{26}(p_{63} + p_{65})\} + p_{43}p_{38}p_{82}\ \{p_{14}p_{25} - p_{15}p_{24} + p_{26}(p_{14}p_{65} - p_{15}p_{64})\}\}]$ (81)

VIII. BUSY PERIOD ANALYSIS OF THE ASSISTANT REPAIRMAN

In steady state, the total fraction of time for which the system in under repair of the assistant repairman is given by

$$B_0^a = N_3 / D_1 \tag{82}$$

Where $N_3 = \mu_6\ p_{26}\ [p_{38}p_{82}(p_{13} + p_{14} + p_{15}) + p_{12}\ (p_{30} + p_{38})]$ (83)

IX. EXPECTED NUMBER OF VISITS BY THE EXPERT REPAIRMAN

In steady state, the number of visits per unit time by the expert is given by

$$V_0^e = N_4 / D_1 \tag{84}$$

where $N_4 = p_{38}p_{82}\ [p_{26}\{p_{26}\{p_{63}(p_{14}p_{45} + p_{15}) - p_{43}(p_{14}p_{65} - p_{15}p_{64}) - p_{13}(p_{65} + p_{64}p_{45})\} - \{p_{15}(p_{64}p_{43} + p_{63}) - p_{45}\ (p_{13}p_{64} - p_{14}p_{63}) - p_{65}(p_{13} + p_{14}p_{43})\}] + (p_{30} + p_{38})\ [(p_{13} + p_{14} + p_{15})\ (1+p_{26}p_{68}p_{82}) + p_{12}\ \{p_{24} + p_{25} + p_{26}\ (p_{63} + p_{64} + p_{65}\}]$ (85)

X. EXPECTED NUMBER OF VISITS BY THE ASSISTANT REPAIRMAN

In steady state, the number of visits per unit time by the assistant repairman is given by

$$V_0^a = N_5 / D_1 \tag{86}$$

Where $N_5 = p_{26}[p_{12}(p_{30} + p_{38}) + p_{38}p_{82}(p_{13} + p_{14} + p_{15})]$ (87)

XI. EXPECTED DISCUSSION TIME

In steady state, the total fraction of the time for which the expert is discussing the nature of failure is given by $DT_0 = N_6 / D_1$ (88)

Where $N_6 = \mu_2[p_{12}(p_{30} + p_{38}) + p_{38}p_{82}(p_{13} + p_{14} + p_{15})]$ (89)

XII. PROFIT ANALYSIS

The expected total profit incurred to the system in steady state is given by

$$P_1 = C_0A_0 - C_1B_0^e - C_2B_0^a - C_3V_0^e - C_4V_o^e - C_5 - C_6\ (DT_0) \tag{90}$$

where

C_0 = revenue per unit up time of the system

C_1 = cost per unit time for which the expert repairman is busy in repairing a failed unit

C_2 = cost per unit time for which the assistant repairman is busy

C_3 = cost per visit of the expert repairman

C_4 = cost per visit of the assistant repairman

C_5 = cost per unit time for the regular repairman

C_6 = cost per unit time for which the expert is busy in discussing the nature of failure of a unit

XIII. COMPARATIVE STUDY BETWEEN THE PRESENT MODEL AND THAT STUDIED BY [6]

In the paper discussed by Taneja and Gupta [6], the assistant/regular repairman is allowed to take his own time to repair till he declares himself unable to complete the repair; and the expert repairman comes immediately as and when the regular/assistant repairman declares himself unable to complete the repair or otherwise required.

Here, in the present paper waiting for the expert is not done beyond certain time i.e. patience time while the assistant tries to complete the repair. The expert repairman comes to the system immediately on the expiry of the patience time or whenever the system becomes inoperable and repairs all the units which fails during his stay at the system.

Let P be the profit incurred to the system for the model discussed by [6].

For making comparative study, let us take $g(t) = \alpha\ e^{-\lambda t}$, $g_e(t) = \alpha_1\ e^{-\alpha 1t}$, $g_a(t) = \alpha_2\ e^{-\alpha 2t}$, $i(t) = \gamma e^{-\gamma t}$, $w(t) = \beta e^{-\beta t}$, $h_2(t) = \beta_2 e^{-\beta 2t}$, and the values of various parameters as mentioned in the following graph which shows the behaviour of the difference of profits with respect to cost per unit discussion time for different values of discussion rate.

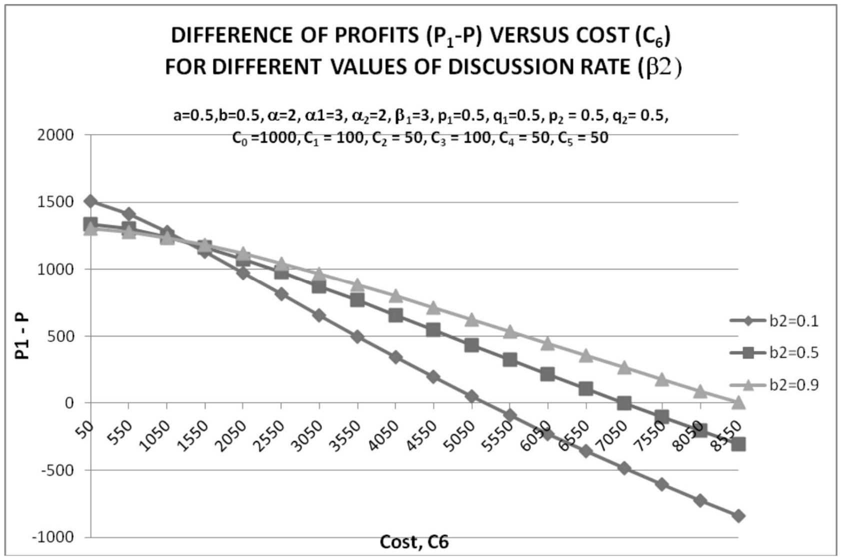

Fig. 2

XIV CONCLUSION:

Following conclusion can be drawn from the graph:

(i) If β_2 = 0.1; P_1-P > 0 or = 0 or< 0 according as C_6 < 5250 or = 5250 or > 5250.
i.e. Model of this present paper is better or worse than Model of [6] according as C_6 < 5250 or > 5250. The two models are equally good if C_6 = 5250.

(ii) If β_2 = 0.5; P_1-P > 0 or = 0 or< 0 according as C_6 < 7058 or = 7058 or > 7058.
i.e. Model of this present paper is better or worse than Model of [6] according as C_6 < 7058 or > 7058. The two models are equally good if C_6 = 7058.

(iii) If β_2 = 0.9; P_1-P > 0 or = 0 or< 0 according as C_6 < 8553 or = 8553 or > 8553.
i.e. Model of this present paper is better or worse than Model of [6] according as C_6 < 8553 or > 8553. The two models are equally good if C_6 = 8553.

This comparative study has been made for the above particular case,. However, the user of such systems may take the values of various parameters as existing for their systems and apply the general results obtained in the present paper and that in [6]. In this way, the cut off points can be obtained to reveal as to which and when one model is better than the other.

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ANALYSIS OF A SINGLE-UNIT SYSTEM WITH DIFFERENT FAILURE MODES BY USING REGENERATIVE POINT GRAPHICAL TECHNIQUE

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ABSTRACT :
The specific word reliability = re + liability means repeated liability because of various breakdowns and failures of a device/system. The reliability engineering constitutes the core of the economy and the development of the infrastructure of a country. For a developing country like India with limited resources, it is all the more important to use its resources in a very optimal way, to achieve higher production by having increased reliability of the systems/equipments. Therefore, for upgrading a system the analysis needs to be done more economically, accurately and quickly without any cumbersome methodology and calculations. The main objective of this paper is to do the reliability analysis of a stochastic system, using Regenerative Point Graphical Technique instead of the regenerative point technique which needs a lot of calculation and simplification work. The analysis is done by using RPGT, of a single unit system which may fail partially before failing totally or it may directly fail totally with the server available on demand and repair done with or without inspection after the system fails totally or partially. As the system may not be fully available in its reduced/partially failed state, so the fuzzy measure of the state is used. The failure, repair and inspection time of the unit are independent and uncorrelated random variables.

Keywords: ***MTSF, Availability, RPGT, Directed Path, Regenerative Point, State, Graph.***

1. INTRODUCTION: Many researchers including Tuteja and Malik [1], Goyal, Rashmi & Ashok Kumar [2], Chander & Bansal [3] have done analysis of stochastic systems (under steady state conditions), using the following formulae of the regenerative point technique:

a) MTSF $= \lim\limits_{s\to 0} \dfrac{1-\tilde{\phi}_0(s)}{s}$

b) Availability $= \lim\limits_{t\to\infty} A_0(t) = \lim\limits_{s\to 0} s.A_0^*(s)$

c) Busy Period of the Server $= \lim\limits_{t\to\infty} B_0(t) = \lim\limits_{s\to 0} s.B_0^*(s)$

d) Expected number of Server's visits/replacements $= \lim\limits_{t\to\infty} \dfrac{V_0(t)}{t} = \lim\limits_{s\to 0} s.\ \tilde{V}_0(s)$

The regenerative point technique is very lengthy and cumbersome as one has to do a lot of calculation and simplification work. To overcome these difficulties, Gupta [4] introduced Regenerative Point Graphical Technique (RPGT). Gupta, Singh, et al [5, 6] and Sarla etal [8] used RPGT to find all the system parameters of vital significance such as mean time to the system failure (MTSF), steady state availability, busy period of the server and expected number of visits by the server and number of replacements etc.

The objective of this paper is to do the reliability/availability analysis, using RPGT introduced by Gupta [4], of a single unit system which has already been discussed and analysed by Chander & Bansal [3], using regenerative point technique. The system may fail partially before failing totally or it may directly fail totally with the server available on demand and repair done with or without inspection after the system fails totally or partially. There is a single server who is called immediately whenever needed. In Model 1, server repairs the unit only at its total failure while in Model 2, he first inspects the total failure unit to see the possibility of its repair or replacement. If repair is possible, he starts it immediately; otherwise the failed unit is replaced by new one. In Model 3, the server repairs the unit at its partial failure and as well as on its total failure, without doing inspection. The failure time distribution of the unit is taken to be negative exponential while the distribution of repair and inspection is arbitrary with different probability density function. Regenerative Point Graphical Technique has been used to calculate the system parameters such as mean time to the system failure (MTSF), steady state availability, busy period of the server and expected number of visits by the server.

2. ASSUMPTIONS, NOTATIONS & SYMBOLS:

2.1. Assumptions: The following assumptions have been made for the analysis of the system:

1. : The system is a single-unit reliability Model in which the unit may fail totally either directly from normal mode or via partial failure.
2. : There is a single server who attends the system immediately on demand and do not leave the system during repairs and after inspection.
3 : The unit works as the new one after repairs.
4. : The distributions of repair and inspections times are arbitrary, while unit has negative exponential distribution of time to the failure.
5. : The system remains operative during repairs of the partially failed unit.
6. : All the random variables are mutually independent.

2.2. Notations: The various notations of regenerative point graphical technique Gupta [4] along with the following notations and symbols have been used in the analysis of the system.

*	:	Laplace Transformation
$q_{i,j}(t)$	:	Probability density function (p.d.f.) of the first passage time from the regenerative state *i* to the regenerative state *j* or to a failed state *j* without visiting any other regenerative state in (0, *t*], given that the system entered the regenerative state *i* at *t* = 0.
$p_{i,j}$	:	The steady state transition probability from the regenerative state *i* to the regenerative state *j* without visiting any other state. $p_{i,j} = q^*_{i,j}(0)$.
$R_i(t)$	:	The reliability that the system in the regenerative state *i* at time *t*, given that the system entered the regenerative state *i* at *t* = 0.
$A_i(t)$	:	Probability that the system is available in up-state '*i*' at time *t*, given that the system entered the regenerative state '*i*' at *t* = 0.
$B_i(t)$	:	Probability that the server is busy doing a particular job at epoch *t*, given that the system entered the regenerative state '*i*' at *t* = 0.
$W_i(t)$	:	Probability that the server is busy doing a particular job at epoch *t* without transiting to any other regenerative state or returning to the regenerative state '*i*' through one or more non-regenerative states, given that the system entered the regenerative state '*i*' at *t* = 0.
η_i	:	Expected waiting time spent while doing a given job, given that the system entered the regenerative state '*i*' at *t* = 0; $\eta_i = W^*_i(0)$.
$\lambda_1/\lambda_2/\lambda$	:	Constant failure rate from normal mode to partial failure mode /partial failure mode to total failure mode /normal mode to total failure mode .
f_i	:	Fuzziness measure of the *i*-state. $f_i = 0$, if '*i*' is a failed state; $f_i = 1$, if '*i*' is an up-state; $f_i \in (0, 1)$ if '*i*' is a reduced/partially failed state.
p/q	:	Probability that the unit is not repairable/ repairable.
g(*t*)/*G*(*t*), $g_1(t)/G_1(t)$	:	Probability density function/cumulative distribution function of the repair-time of totally failed unit and partially failed unit. $\overline{G}_i(t) = 1 - G_i(t)$.
h(*t*)/*H*(*t*)	:	Probability density function/cumulative distribution function of the inspection times. $\overline{H}(t) = 1 - H(t)$.
FUi/ FUr	:	Unit is totally failed and under inspection/under repair.
O/PF/PUr	:	Unit is in normal mode/ partially failed/partially-failed unit under repair.

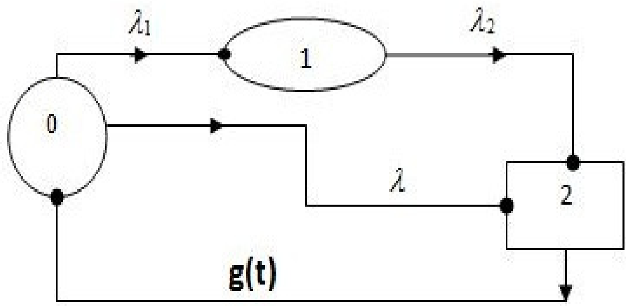
2.3. Symbols:

<table>
<tr><th>State</th><th>Symbol</th><th colspan="2">Model 1</th><th colspan="2">Model 2</th><th colspan="2">Model 3</th></tr>
<tr><td>Regenerative State</td><td>●</td><td colspan="2">0,1,2</td><td colspan="2">0,1,2,3</td><td colspan="2">0,1,2</td></tr>
<tr><td>Up-State</td><td>○</td><td colspan="2">0</td><td colspan="2">0</td><td colspan="2">0</td></tr>
<tr><td>Down State</td><td>⬭</td><td>1</td><td>PF</td><td>1</td><td>PF</td><td>1</td><td>PUr</td></tr>
<tr><td rowspan="2">Failed State</td><td rowspan="2">▭</td><td rowspan="2">2</td><td rowspan="2">FUr</td><td>2</td><td>FUi</td><td rowspan="2">2</td><td rowspan="2">FUr</td></tr>
<tr><td>3</td><td>FUr</td></tr>
</table>

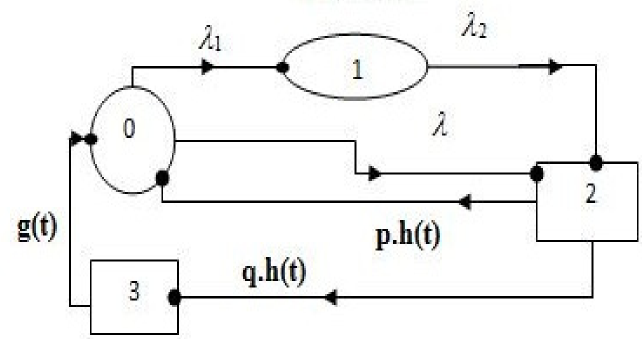
3. STATE TRANSITION DIAGRAM OF THE SYSTEM:

The state transition diagrams of Model 1, 2 & 3 are as under:

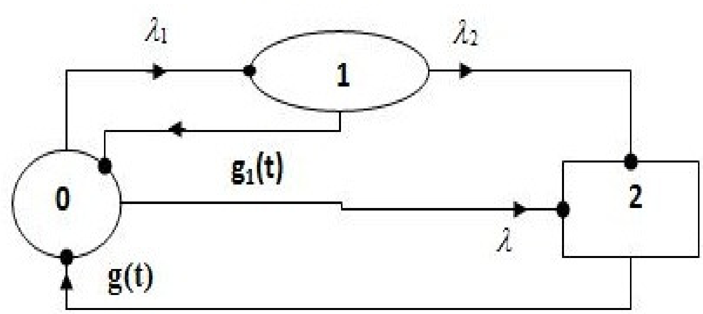
Model 1



Model 2



Model 3



PATHS

Model 1

Number of Vertices: 3, Number of Edges (4)
X: 0,Y: 1; X: 1,Y: 2;
X: 2,Y: 0; X: 0,Y: 2.

Initial State	Final State	Paths
0	0	0→1→2→0
		0→2→0
0	1	0→1
0	2	0→1→2
		0 →2
1	0	1→2→0
1	1	1→2→0→1
1	2	1→2
2	0	2→0
2	1	2→0→1
2	2	2→0→1→2
		2→0→2

Model 2

Number of Vertices : 4, Number of Edges (6)
X: 0,Y: 1; X: 0,Y: 2; X: 1,Y: 2;
X: 2,Y: 0; X: 2,Y: 3; X: 3,Y: 0

Initial State	Final State	Paths
0	0	0→1→2→0
		0→1→2→3→0
		0→2→0
		0→2→3→0
0	1	0→1
0	2	0→1→2
		0→2
0	3	0→1→2→3
		0→2→3
1	0	1→2→ 0
		1→2→3→0
1	1	1→2→0→1
		1→2→3→0→1
1	2	1→2
1	3	1→2→3
2	0	2→0
		2→3→0
2	1	2→0→1
		2→3→0→1
2	2	2→0→1→2
		2→0→2
		2→3→0→1→2
		2→3→0→2
2	3	2→3
3	0	3→0
3	1	3→0→1
3	2	3→0→1→2
		3→0→2
3	3	3→0→1→2→3
		3→0→2→3

Model 3

Number of Vertices :3, Number of Edges (5)
X: 0,Y: 1;X: 0,Y: 2;X: 1,Y: 2;X: 1,Y: 0;X: 2,Y: 0

Initial State	Final State	Paths
0	0	0→1→2→0
		0→1→0; 0→2→0
0	1	0→1
0	2	0→1→2; 0→2
1	0	1→2→0; 1→0
1	1	1→2→0→1
		1→0→1
1	2	1→2; 1→0→2
2	0	2→0
2	1	2→0→1
2	2	2→0→1→2
		2→0→2

4. STEADY STATE TRANSITION PROBABILITES AND MEAN SOJOURN –TIMES:

Model 1			
$q_{i,j}(t)$	$p_{i,j} = q_{i,j}^{*}(0)$	$R_i(t)$	$\mu_i = R_i^{*}(0)$
$q_{0,1}(t) = \lambda_1 e^{-(\lambda+\lambda_1).t}$ $q_{0,2}(t) = \lambda e^{-(\lambda+\lambda_1).t}$	$p_{0,1} = \dfrac{\lambda_1}{\lambda+\lambda_1}$ $p_{0,2} = \dfrac{\lambda}{\lambda+\lambda_1}$	$R_0(t)$ $= e^{-(\lambda+\lambda_1).t}$	$\mu_0 = \dfrac{1}{\lambda+\lambda_1}$
$q_{1,2}(t) = \lambda_2 e^{-(\lambda_2).t}$	$p_{1,2} = 1$	$R_1(t) = e^{-(\lambda_2).t}$	$\mu_1 = \dfrac{1}{\lambda_2}$
$q_{2,0}(t) = g(t)$	$p_{2,0} = g^*(0)$	$R_2(t) = \overline{G}(t)$	$\mu_2 = -g^{*\prime}(0)$
Model 2			
$q_{0,1}(t) = \lambda_1 e^{-(\lambda+\lambda_1).t}$ $q_{0,2}(t) = \lambda e^{-(\lambda+\lambda_1).t}$	$p_{0,1} = \dfrac{\lambda_1}{\lambda+\lambda_1}$ $p_{0,2} = \dfrac{\lambda}{\lambda+\lambda_1}$	$R_0(t)$ $= e^{-(\lambda+\lambda_1).t}$	$\mu_0 = \dfrac{1}{\lambda+\lambda_1}$
$q_{1,2}(t) = \lambda_2 e^{-(\lambda_2).t}$	$p_{1,2} = 1$	$R_1(t) = e^{-(\lambda_2).t}$	$\mu_1 = \dfrac{1}{\lambda_2}$
$q_{2,0}(t) = ph(t)$ $q_{2,3}(t) = qh(t)$	$p_{2,0} = ph^*(0)$ $p_{2,3} = qh^*(0)$	$R_2(t) = \overline{H}(t)$	$\mu_2 = -h^{*\prime}(0)$
$q_{3,0}(t) = g(t)$	$p_{3,0} = g^*(0)$	$R_3(t) = \overline{G}(t)$	$\mu_3 = -g^{*\prime}(0)$
Model 3			
$q_{0,1}(t) = \lambda_1 e^{-(\lambda+\lambda_1).t}$ $q_{0,2}(t) = \lambda e^{-(\lambda+\lambda_1).t}$	$p_{0,1} = \dfrac{\lambda_1}{\lambda+\lambda_1}$ $p_{0,2} = \dfrac{\lambda}{\lambda+\lambda_1}$	$R_0(t)$ $= e^{-(\lambda+\lambda_1).t}$	$\mu_0 = \dfrac{1}{\lambda+\lambda_1}$
$q_{1,2}(t) = \lambda_2 e^{-(\lambda_2).t}\,\overline{G}_1(t)$ $q_{1,0}(t) = g_1(t)\, e^{-(\lambda_2).t}$	$p_{1,2} = 1 - g_1^*(\lambda_2)$ $p_{1,0} = g_1^*(\lambda_2)$	$R_1(t) = e^{-(\lambda_2).t}\,\overline{G}_1(t)$	$\mu_1 = \dfrac{1-g_1^*(\lambda_2)}{\lambda_2}$
$q_{2,0}(t) = g(t)$	$p_{2,0} = g^*(0) = 1$	$R_2(t) = \overline{G}(t)$	$\mu_2 = -g^{*\prime}(0)$

5. EVALUATION OF THE PARAMETERS USING RPGT:

5.1 Mean Time to System Failure (MTSF): The mean time to system failure under steady state conditions is given by using RPGT:

$$\text{MTSF} = \left[\sum_{i,s_r}\left\{\frac{\{pr(0\xrightarrow{s_r(sff)}i)\}.\mu_i}{\prod_{k_1\neq 0}\{1-\sum pr(k_1-\overline{cycle})\}}\right\}\right] \div \left[1-\sum_{s_r}\left\{\frac{\{pr(0\xrightarrow{s_r(sff)}0)\}}{\prod_{k_2\neq 0}\{1-\sum pr(k_2-\overline{cycle})\}}\right\}\right]$$

The regenerative un-failed states to which the system can transit before entering any failed state are:

$i = 0, 1$ (**Model 1, 2 & 3**)

For Models 1. & 2. : k_1 = nil; k_2 = nil MTSF = $[p_{0,0}\mu_0 + p_{0,1}\mu_1] \div [1-0]$

$= [\mu_0 + p_{0,1}\mu_1]$

For Model 3. : k_1 = nil; $k_2 = 1$ MTSF = $[p_{0,0}\mu_0 + p_{0,1}\mu_1] \div [1-p_{0,1}\, p_{1,0}]$

$= [\mu_0 + p_{0,1}\mu_1] \div [1-p_{0,1}\, p_{1,0}]$

5.2 Steady State Availability of the System: Total fraction of time for which the system is available (under steady state conditions) is given by using RPGT:

$$A_0 = \left[\sum_{j,s_r}\left\{\frac{\{pr(0\xrightarrow{s_r}j)\}f_j.\mu_j}{\prod\limits_{k_1\neq 0}\{1-\sum pr(k_1-cycle)\}}\right\}\right] \div \left[\sum_{i,s_r}\left\{\frac{\{pr(0\xrightarrow{s_r}i)\}.\mu_i^1}{\prod\limits_{k_2\neq 0}\{1-\sum pr(k_2-cycle)\}}\right\}\right]$$

The regenerative states at which system is in up-state/down or partially failed states are: j = 0 with $f_0 = 1$; $j = 1$ with $f_1 \in (0, 1)$. k_1 = nil; k_2 = nil (**Model 1, 2 & 3**).

For Model 1. : $f_0 = 1$, $\mu_i = \mu_i'$ and the regenerative states are: $i = 0, 1$ & 2.

$$A_0 = [p_{0,0}\mu_0 + p_{0,1}\mu_1 f_1] \div [p_{0,0}\mu_0 + p_{0,1}\mu_1 + (p_{0,1}p_{1,2} + p_{0,2})\mu_2]$$

$$= [\mu_0 + p_{0,1}\mu_1 f_1] \div [\mu_0 + p_{0,1}\mu_1 + \mu_2]$$

For Model 2. : $f_0 = 1$, $\mu_i = \mu_i'$ and the regenerative states are: $i = 0, 1, 2$ & 3.

$$A_0 = [p_{0,0}\mu_0 + p_{0,1}\mu_1 f_1] \div [p_{0,0}\mu_0 + p_{0,1}\mu_1 + (p_{0,1}p_{1,2} + p_{0,2})\mu_2 + (p_{0,1}p_{1,2} + p_{0,2})p_{2,3}\,\mu_3]$$

$$= [\mu_0 + p_{0,1}\mu_1 f_1] \div [\mu_0 + p_{0,1}\mu_1 + \mu_2 + p_{2,3}\,\mu_3]$$

For Model 3. : $f_0 = 1$, $\mu_i = \mu_i'$ and the regenerative states are: $i = 0, 1$ & 2.

$$A_0 = [p_{0,0}\mu_0 + p_{0,1}\mu_1 f_1] \div [p_{0,0}\mu_0 + p_{0,1}\mu_1 + (p_{0,1}p_{1,2} + p_{0,2})\mu_2]$$

$$= [\mu_0 + p_{0,1}\mu_1 f_1] \div [\mu_0 + p_{0,1}\mu_1 + (p_{0,1}p_{1,2} + p_{0,2})\mu_2]$$

5.3 Busy Period of the Server:

The busy period of the server is given by using RPGT:

$$B_0 = \left[\sum_{j,s_r}\left\{\frac{\{pr(0\xrightarrow{s_r}j)\}.\eta_j}{\prod\limits_{k_1\neq 0}\{1-\sum pr(k_1-cycle)\}}\right\}\right] \div \left[\sum_{i,s_r}\left\{\frac{\{pr(0\xrightarrow{s_r}i)\}.\mu_i^1}{\prod\limits_{k_2\neq 0}\{1-\sum pr(k_2-cycle)\}}\right\}\right]$$

The regenerative states where the server is busy doing any job like repair/replacement/inspection are:

$j = 2$ (**Model 1**), $j = 2, 3$ (**Model 2**), $j = 1, 2$ (**Model 3**). k_1= nil; k_2 = nil (**Model 1, 2 & 3**)

For Model 1. : $\eta_2 = \mu_2$, $\mu_i = \mu_i'$ and the regenerative states are : $i = 0, 1$ & 2.

$$B_0 = [(p_{0,1}p_{1,2} + p_{0,2})\mu_2)] \div [p_{0,0}\mu_0 + p_{0,1}\mu_1 + (p_{0,1}p_{1,2} + p_{0,2})\mu_2]$$

$$= [\mu_2] \div [\mu_0 + p_{0,1}\mu_1 + \mu_2]$$

For Model 2. : $\eta_2 = \mu_2, \eta_3 = \mu_3$, $\mu_i = \mu_i'$ and the regenerative states are: i = 0, 1, 2 & 3.

$$B_0 = [(p_{0,1}p_{1,2} + p_{0,2})\mu_2) + [(p_{0,1}p_{1,2} + p_{0,2})p_{2,3}\,\mu_3)] \div$$
$$[p_{0,0}\mu_0 + p_{0,1}\mu_1 + (p_{0,1}p_{1,2} + p_{0,2})\mu_2 + (p_{0,1}p_{1,2} + p_{0,2})p_{2,3}\,\mu_3)]$$
$$=[\mu_2 + p_{2,3}\,\mu_3)] \div [\mu_0 + p_{0,1}\mu_1 + \mu_2 + p_{2,3}\,\mu_3)]$$

For Model 3. : $\eta_1 = \mu_1$, $\eta_2 = \mu_2$, $\mu_i = \mu_i'$ and the regenerative states are: i = 0, 1 & 2.

$$B_0 = [p_{0,1}\mu_1 + (p_{0,1}p_{1,2} + p_{0,2})\mu_2)] \div [p_{0,0}\mu_0 + p_{0,1}\mu_1 + (p_{0,1}p_{1,2} + p_{0,2})\mu_2]$$
$$= [p_{0,1}\mu_1 + (p_{0,1}p_{1,2} + p_{0,2})\mu_2)] \div [\mu_0 + p_{0,1}\mu_1 + (p_{0,1}p_{1,2} + p_{0,2})\mu_2]$$

5.4 Expected Number of Visit by the Server:

The expected number of visits of the server/replacements can be calculated by using the RPGT:

$$V_0 = \left[\sum_{j,s_r}\left\{\frac{\{pr(0\xrightarrow{s_r} j)\}}{\prod\limits_{k_1\neq 0}\{1-\sum pr(k_1\text{−}cycle)\}}\right\}\right] \div \left[\sum_{i,s_r}\left\{\frac{\{pr(0\xrightarrow{s_r} i)\}.\mu_i^1}{\prod\limits_{k_2\neq 0}\{1-\sum pr(k_2\text{−}cycle)\}}\right\}\right]$$

The regenerative states where the server visits afresh may be after exiting and then again entering the system are : j = 2 (**Model 1 & 2**), j =1,2 (**Model 3**) ; k_1= nil , k_2 = nil (**Model 1,2 & 3**).

For Model 1. : $\mu_i = \mu_i'$ and the regenerative states are: i = 0, 1 & 2.

$$V_0 = [(p_{0,1}p_{1,2} + p_{0,2})] \div [\, p_{0,0}\mu_0 + p_{0,1}\mu_1 + (p_{0,1}p_{1,2} + p_{0,2})\mu_2]$$
$$= [\,\mu_0 + p_{0,1}\mu_1 + \mu_2]^{-1}$$

For Model 2. : $\mu_i = \mu_i'$ and the regenerative states are: i = 0, 1, 2 & 3.

$$V_0 = [(p_{0,1}p_{1,2} + p_{0,2})]$$
$$\div[p_{0,0}\mu_0 + p_{0,1}\mu_1 + (p_{0,1}p_{1,2} + p_{0,2})\mu_2 + (p_{0,1}p_{1,2} + p_{0,2})p_{2,3}\,\mu_3)]$$
$$= 1 \div [\mu_0 + p_{0,1}\mu_1 + \mu_2 + p_{2,3}\,\mu_3)]$$

For Model 3. : $\mu_i = \mu_i'$ and the regenerative states are: i = 0, 1 & 2.

$$V_0 = [(p_{0,1} + p_{0,2})] \div [p_{0,0}\mu_0 + p_{0,1}\mu_1 + (p_{0,1}p_{1,2} + p_{0,2})\mu_2)]$$
$$= [(p_{0,1} + p_{0,2})] \div [\mu_0 + p_{0,1}\mu_1 + (p_{0,1}p_{1,2} + p_{0,2})\mu_2)]$$

6. CONCLUSION:

The results obtained in Sec. 5, using the RPGT are the same (on taking $f_1 = 1$) as are obtained by Chander & Bansal [3], by using regenerative-point technique. Clearly, the key parameters of the system are obtained very easily and quickly without writing any state equations and without doing any cumbersome calculations etc.

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ON EINSTEIN'S PRIORITIZED WEIGHTED AVERAGE OPERATORS OF TYPE s, s ≠ 0 UNDER THE ENVIRONMENT OF INTUITIONISTIC FUZZY SET THEORY

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ABSTRACT :
In this paper, Einstein's prioritized weighted averaging operators of type s, $s \neq 0$ under the intuitionistic fuzzy set theoretic environment have been discussed. Some properties along related theorems have been verified.

[1] INTRODUCTION

Atanassov [1] generalized Zadeh's [13] fuzzy set theory by introducing the degree of membership function, non-membership function and the hesitation degree function in 1986 and called it intuitionistic fuzzy set theory. Fuzzy sets are intuitionistic fuzzy set but the converse is not necessarily true. Recently, a lot of applications of IFSs have been made in management oriented decisions science. Decision makers have considered various angles such as multi-criteria, multi-attribute, multi-group, information aggregation-geometric aggregation operations, weighted geometric operators, ordered weighted geometric operators, hybrid geometric operators and many others.

Recently Xia and Xu [5, 6] developed a number of generalized intuitionistic fuzzy aggregation operators such as generalized intuitionistic fuzzy point weighted averaging (GIFPWA) operators, the generalized intuitionistic fuzzy point ordered weighted averaging (GIFPOWA) operator and the generalized intuitionistic fuzzy point hybrid averaging (GIFPHA) operator and studied their properties. Wei [4] introduced some induced intuitionistic fuzzy geometric aggregation operators and applied them in group decision making problems. Zeng and Su [14] proposed the intuitionistic fuzzy ordered weighted distance (IFOWP) operator to aggregate the intuitionistic fuzzy information. The dynamic intuitionistic fuzzy multiple-attribute decision making prolems were introduced by Xu and Yager [8]. They also developed dynamic intuitionistic fuzzy weighted averaging (DIFWA) operator to aggregate dynamic intuitionistic fuzzy information. Wei [4] also handled the dynamic intuitionistic fuzzy weighted geometric (DIFWG) operator and applied to dynamic multiple-attribute decision making with intuitionistic fuzzy information.

Recently Wang and Liu [3] introduced an IFSs Einstein sum, product, exponentiation and developed some new intuitionistic fuzzy aggregation operators – intuitionisitc fuzzy Einstein weighted average (IEEWA) operator and ordered weighted average (IFEOWA) operator.

They also established some necessary properties such as commutativity, idempotency, monotonicity and developed a decision making method for solving multi-attribute decision making problems under intuitionistic fuzzy environment. Xu et al. [7] introduced induced intuitionistic fuzzy Einstein ordered weighted averaging (I-IFEDWA) operator for aggregating intuitionistic fuzzy information and applied to multi-attribute group decision making. Verma and Sharma [2] extended the studies further for prioritized weighted average operators and applied to multi-attribute group decision making under intuitionistic fuzzy environment. In the present communication, we have discussed Einstein's prioritized weighted average operators of type s, $s \neq 0$ under the environment of intuitionistic fuzzy set theory along with some properties and related theorems.

[2] SOME BASIC CONCEPTS

Let us have some basic definitions, which we will frequently utilize in the present communication.

- **Intuitionistic Fuzzy Set**

Definition: 1: IFS – Intuitionistic fuzzy set.

Let $A = \{\mu_A(x), \nu_s(x);\ x \in X\}$ (1)

A being a discrete universe of discourse

$$\left.\begin{array}{l} X = (x_1, x_2, ..., x_n) \\ \mu_A : X \to [0, 1] \\ \nu_A : X \to [0,1] \end{array}\right\} \quad (2)$$

with the condition $0 \le \mu_A(x) + \nu_A(x) \le 1$. (3)

and $\pi_A = 1 - \mu_A(x) - \nu_A(x)$, (4)

is the degree of hesitation or intuitionistic index of x in A.

- **Intuitionistic Fuzzy Number**

Xu and Yagar [8] and Xu [9] considered the pair $(\mu_A(x), \nu_A(x))$ as intuitionistic fuzzy number (IFN) and denoted it by $\alpha = (\mu_\alpha, \nu_\alpha)$ (5)

- **Einstein Operations on IFNs [19, 20]**

Definition 2: Let $\alpha_1 = (\mu_{\alpha_1}, \nu_{\alpha_1})$, $\alpha_2 = (\mu_{\alpha_2}, \nu_{\alpha_2})$ be two intuitionistic fuzzy numbers and $\lambda > 0$, then the following Einstein operations on α_1 and α_2 are defined as

(i) $$\alpha_1 \oplus \alpha_2 = \left(\left(\frac{\mu_{\alpha_1} + \mu_{\alpha_2}}{1 + \mu_{\alpha_2}\mu_{\alpha_2}} \right), \frac{\nu_{\alpha_1} + \nu_{\alpha_2}}{1 + (1 - \nu_{\alpha_1})(1 - \nu_{\alpha_2})} \right) \quad (6)$$

(ii) $$\alpha_1 \otimes \alpha_2 = \left(\frac{\mu_{\alpha_1} + \mu_{\alpha_2}}{1 + (1 - \mu_{\alpha_1})(1 - \mu_{\alpha_2})}, \frac{\nu_{\alpha_1} + \nu_{\alpha_2}}{1 + \nu_{\alpha_1}\nu_{\alpha_2}} \right) \quad (7)$$

(iii) $$\lambda . e^{\alpha_1} = \left(\frac{(1 + \mu_{\alpha_1})^\lambda - (1 - \mu_{\alpha_1})^\lambda}{(1 + \mu_{\alpha_1})^\lambda + (1 - \mu_{\alpha_1})^\lambda}, \frac{\lambda \nu_{\alpha_1}^\lambda}{(2 - \nu_{\alpha_1})^\lambda + \nu_{\alpha_1}^\lambda} \right) \quad (8)$$

(iv) $$(\alpha_1)^\lambda e^\lambda = \left(\frac{\lambda \mu_{\alpha_1}^\lambda}{(2 - \mu_{\alpha_1})^\lambda + \mu_{\alpha_1}^\lambda}, \frac{1 + \nu_{\alpha_1}) + (1 - \nu_{\alpha_1})^\lambda}{(1 + \nu_{\alpha_1})^\lambda + (1 - \nu_{\alpha_1})^\lambda} \right) \quad (9)$$

- **Prioritized Weighted Average Operator of Type s, s ≠ 0**

Definition: Let $G^s = \{G_1^s, G_2^s, ..., G_n^s\}$, be a collection of attributes and let there be a prioritization between the attributes by a linear ordering $G_1^s > G_s^s > G_3^s .... > G_n^s$, indicating the attribute G_j^s has a high priority than G_k^s, and satisfies $G_j^s(X) \in [0, 1]$ if

$$PWA(G_1^s(x), G_2^s(x), ..., G_n^s(x)) = \sum_{j=1}^{n} \frac{T_j^s}{\sum_{i=1}^{n} T_j^s} G_j(x), \qquad s > 0 \quad (10)$$

where $T_j^s = \prod_{k=1}^{i-1} G_k^s(x), \;\; j = 2,3,....,n, T_1^s = 1$ then PWA is called the prioritized weighted average operator of type *s, s* ≠ 0.

- **Intuitionistic Fuzzy Einstein Prioritized Weighted Average Operators of Type s, s ≠ 0**

Definition: Let the set of intuitionistic fuzzy numbers $\alpha_j = (\mu_{\alpha_j}, \nu_{\alpha_j})$, j = 1, 2, ...n, be the intuitionistic fuzzy Einstein prioritized weighted average (IFEPWA) operator of type *s, s* ≠ 0 is defined as follows:

IFEPWA $(\alpha_1^s, \alpha_2^s, ..., \alpha_n^s)$

$$= \left(\overset{n}{\underset{j=1}{\oplus}} \frac{T_j^s}{\sum_{j=1}^{n} T_j^s} . e^{\alpha_j} \right) \tag{11}$$

$$= \left(\frac{T_1^s}{\sum_{j=1}^{n} \alpha_j} . e^{\alpha_1} \oplus \frac{T_2^s}{\sum_{j=1}^{n} \alpha_j} . e^{\alpha_2} \oplus .... \oplus \frac{T_n^s}{\sum_{j=1}^{n} \alpha_j} . e^{\alpha_n} \right) \tag{12}$$

where

$$T_j^s = \prod_{k=1}^{j-1} S(\alpha_k^s), \qquad j = 2,3,....,n, T_1^s = 1 \tag{13}$$

and $S(\alpha_k^s)$ is the score of $\alpha_k^s = (\mu_{2k}^s, \nu_{2k}^s)$ and $S(\alpha)^s = \mu_\alpha^s - \nu_\alpha^s$. $S(\alpha)^s \in [-1, 1]$ (14)

[3] Aggregation an Intuitionistic Number

To show that the aggregation value is an intuitionistic fuzzy number (IFN).

Theorem 1. Let $\alpha_j^s = (\mu_{2j}^s, \nu_{\alpha j}^s)$, j = 1, 2, 3, ...,n be the set of intuitionistic fuzzy numbers, of type *s, s* ≠ 0, then the aggregation value by using the (IFEPWA) operator of type s, s ≠ 0 is also an intuitionistic fuzzy number (IFN) of type *s, s* ≠ 0,

IFEPWA $(\alpha_1^s, \alpha_2^s, ..., \alpha_n^s)$

$$= \left(\frac{T_1^s}{\sum_{i=1}^{n} T_j^s} . e^{\alpha_1^s} \oplus \frac{T_2^s}{\sum_{j=1}^{n} T_j^s} . e^{\alpha_2^s} \oplus .... \oplus \frac{T_n^s}{\sum_{j=1}^{n} T_j^s} . e^{\alpha_n^s} \right) \tag{15}$$

$$\left[\frac{\prod_{j=1}^{n} (1+\mu_{\alpha_j}^s) \frac{T_j^s}{\sum_{i=1}^{n} T_j^s} - \prod_{j=1}^{n} (1+\mu_{\alpha_j}^s) \frac{T_j^s}{\sum_{i=1}^{n} T_j^s}}{\prod_{j=1}^{n} (1+\mu_{\alpha_j}^s) \frac{T_j^s}{\sum_{j=1}^{n} T_j^s} + \prod_{j=1}^{n} (1-\mu_{\alpha_j}^s) \frac{T_j^s}{\sum_{i=1}^{n} T_j^s}}, \right.$$

$$\frac{2\prod_{j=1}^{n} \nu_{\alpha_j}^{s} \frac{T_j^s}{\sum_{i=1}^{n} T_j^s}}{\prod_{j=1}^{n} (2-\nu_{\alpha_j}^{s})\frac{T_j^s}{\sum_{j=1}^{n} T_j^s} + \prod_{j=1}^{n} \nu_{\alpha_j}^{s} \frac{T_j^s}{\sum_{i=1}^{n} T_j^s}} \tag{16}$$

where

$$T_j^s = \prod_{k=1}^{j-1} S^*(\alpha_k^s), \qquad j = 2,3,....n,\ T_1 = 1 \ \text{ and } \ S^*(\alpha_\alpha^s) \tag{17}$$

is the score function of type s, s ≠ 0, of $\alpha_k^s = (\mu_{\alpha_k}^s, \nu_{\alpha_k}^s)$.

Proof: We prove the result by mathematical induction method. Let n = 2, then for α_1 and α_2 vide Einstein operation laws of IFNs, we have

$$\frac{T_1^s}{\sum_{j=1}^{2} T_j^s} e^{\alpha_1^s}$$

$$= \left\{ \frac{(1+\mu_{\alpha_1}^s)\frac{T_1^s}{T_1^s+T_2^s} - (1-\mu_{\alpha_1}^s)\frac{T_1^s}{T_1^s+T_2^s}}{(1+\mu_{\alpha_1}^s)\frac{T_1^s}{T_1^s+T_2^s} + (1-\mu_{\alpha_1}^s)\frac{T_1^s}{T_1^s+T_2^s}}, \frac{2\nu_{\alpha_1}^s \frac{T_1^s}{T_1^s+T_2^s}}{(1-\nu_{\alpha_1}^s)\frac{T_1^s}{T_1^s+T_2^s} + \nu_{\alpha_1}^s \frac{T_1^s}{T_1^s+T_2^s}} \right\} \tag{18}$$

and $\frac{T_1^s}{T_1^s+T_2^s}.e^{\alpha_2^s}$

$$= \left\{ \frac{(1+\mu_{\alpha_2}^s)\frac{T_2^s}{T_1^s+T_2^s} - (1-\mu_{\alpha_2}^s)\frac{T_2^s}{T_1^s+T_2^s}}{(1+\mu_{\alpha_2}^s)\frac{T_2^s}{T_1^s+T_2^s} + (1-\mu_{\alpha_2}^s)\frac{T_2^s}{T_1^s+T_2^s}}, \frac{2\nu_{\alpha_2}^s \frac{T_2^s}{T_1^s+T_2^s}}{(1-\nu_{\alpha_2}^s)\frac{T_2^s}{T_1^s+T_2^s} + \nu_{\alpha_2}^s \frac{T_2^s}{T_1^s+T_2^s}} \right\} \tag{19}$$

By definition (12)

IFEPWA (α_1^s, α_2^s)

$$= \left(\frac{T_1^s}{T_1^s+T_2^s}.e^{\alpha_1^s} \oplus \frac{T_2^s}{T_1^s+T_2^s} e^{\alpha_2^s} \right)$$

$$= \left\{ \frac{(1+\mu_{\alpha_1}^s)\frac{T_1^s}{T_1^s+T_2^s}(1+\mu_{\alpha_1}^s)\frac{T_2^s}{T_1^s+T_2^s} - (1-\mu_{\alpha_1}^s)\frac{T_1^s}{T_1^s+T_2^s}(1-\mu_{\alpha_2}^s)\frac{T_2^s}{T_1^s+T_2^s}}{(1+\mu_{\alpha_1}^s)\frac{T_1^s}{T_1^s+T_2^s}(1+\mu_{\alpha_2}^s)\frac{T_1^s}{T_1^s+T_2^s} + (1-\mu_{\alpha_1}^s)\frac{T_1^s}{T_1^s+T_2^s}(1-\mu_{\alpha_2}^s)\frac{T_2^s}{T_1^s+T_2^s}}, \right.$$

$$\left. \frac{2\nu_{\alpha_1}^s \frac{T_1^s}{T_1^s+T_2^s} 2\nu_{\alpha_2}^s \frac{T_2^s}{T_1^s+T_2^s}}{(1-\nu_{\alpha_1}^s)\frac{T_1^s}{T_1^s+T_2^s}(1-\nu_{\alpha_2}^s)\frac{T_2^s}{T_1^s+T_2^s} + \nu_{\alpha_1}^s \frac{T_1^s}{T_1^s+T_2^s} \nu_{\alpha_2}^s \frac{T_2^s}{T_1^s+T_2^s}} \right\}$$

$$=\left\{\frac{\sum_{i=1}^{2}(1+\mu_{\alpha_j}^{s})\frac{T_j^s}{\sum_{j=1}^{2}}-\sum_{j=1}^{2}(1-\mu_{\alpha_i}^{s})\frac{T_j^s}{\sum_{j=1}^{2}T_j^s}}{\prod_{j=1}^{2}(1+\mu_{\alpha_i}^{s})\frac{T_j^s}{\sum_{j=1}^{2}T_j^s}+\prod_{j=1}^{2}(1-\mu_{\alpha_j}^{s})\frac{T_j^s}{\sum_{j=1}^{2}T_j^s}},\right.$$

$$\left.\frac{2\prod_{j=1}^{2}\nu_{\alpha_j}^{s}\frac{T_j^s}{\sum_{j=1}^{2}T_j^s}}{\prod_{j=1}^{2}(2-\nu_{\alpha_2}^{s})\frac{T_j^s}{\sum_{j=1}^{2}T_j^s}+\sum_{j=1}^{2}\nu_{\alpha_j}^{s}\frac{T_1^s}{\sum_{j=1}^{2}T_j^s}}\right\} \tag{20}$$

Hence the result (16) holds for $n = 2$.

Now suppose (16) holds for $n = k$, that is IFEPWA $(\alpha_1^s, \alpha_1^s, \ldots., \alpha_k^s)$

$$=\left(\frac{T_1^s}{\sum_{j=1}^{K}T_j^s}.e^{\alpha_1^s}\oplus\frac{T_2^s}{\sum_{j=1}^{K}T_j^s}.e^{\alpha_2^s}\oplus\ldots..\oplus\frac{T_k^s}{\sum_{j=1}^{K}T_j^s}.e^{\alpha_k^s}\right)$$

$$=\left\{\frac{\prod_{j=1}^{k}(1+\mu_{\alpha_j}^{s})\frac{T_j^s}{\sum_{j=1}^{k}T_j^s}-\prod_{j=1}^{k}(1-\mu_{\alpha_j}^{s})\frac{T_j^s}{\sum_{j=1}^{k}T_j^s}}{\prod_{j=1}^{k}(1+\mu_{\alpha_j}^{s})\frac{T_j^s}{\sum_{j=1}^{k}T_j^s}+\prod_{j=1}^{k}(1-\mu_{\alpha_j}^{s})\frac{T_j^s}{\sum_{j=1}^{k}T_j^s}}\right.$$

$$\left.\frac{2\prod_{j=1}^{k}\nu_{\alpha_j}^{s}\frac{T_j^s}{\sum_{j=1}^{k}T_j^s}}{\prod_{j=1}^{k}(2-\nu_{\alpha_j}^{s})\frac{T_j^s}{\sum_{j=1}^{k}T_j^s}+\prod_{j=1}^{k}\nu_{\alpha_j}^{s}\frac{T_j^s}{\sum_{j=1}^{k}T_j^s}}\right\} \tag{21}$$

Now by induction method, let $n = k+1$ by Einstein's operation laws of the IFNs, we have IFEPWA $(\alpha_1^s, \alpha_2^s, \ldots., \alpha_{k+1}^s)$

$$=\left(\frac{T_1^s}{\sum_{j=1}^{k+1}T_j^s},e^{\alpha_1^s}\oplus\frac{T_2^s}{\sum_{j=1}^{k+1}T_{j=1}^s}e^{\alpha_2^s}\ldots..\oplus\frac{T_k^s}{\sum_{j=1}^{k}T_j^s}e^{\alpha_k^s}\right)\oplus\frac{T_{k+1}^s}{\sum_{j=1}^{k+1}T_j^s}e^{\alpha_{k+1}^s}$$

$$
=\left\{\frac{\prod_{j=1}^{k}(1+\mu_{\alpha_j}^{s})^{\frac{T_j^s}{\sum_{j=1}^{k}T_j^s}}-\prod_{j=1}^{k}(1-\mu_{\alpha_j}^{s})^{\frac{T_j^s}{\sum_{j=1}^{k}T_j^s}}}{\prod_{j=1}^{k}(1+\mu_{\alpha_j}^{s})^{\frac{T_j^s}{\sum_{j=1}^{k}T_j^s}}+\prod_{j=1}^{k}(1-\mu_{\alpha_j}^{s})^{\frac{T_j^s}{\sum_{j=1}^{k}T_j^s}}}\right.
$$

$$
\left.\frac{2\prod_{j=1}^{k}\nu_{\alpha_j}^{s\,\frac{T_j^s}{\sum_{j=1}^{k}T_j^s}}}{\prod_{j=1}^{k}(2-\nu_{\alpha_j}^{s})^{\frac{T_j^s}{\sum_{j=1}^{k}T_j^s}}+\prod_{j=1}^{k}\nu_{\alpha_j}^{s\,\frac{T_j^s}{\sum_{j=1}^{k}T_j^s}}}\right\}
$$

$$
\oplus\left\{\frac{(1+\mu_{\alpha_{k+1}}^{s})^{\frac{T_{k+1}}{\sum_{j=1}^{k+1}T_j^s}}-(1-\mu_{\alpha_{k+1}}^{s})^{\frac{T_{k+1}^s}{\sum_{j=1}^{k+1}T_j^s}}}{(1+\mu_{\alpha_{k+1}}^{s})^{\frac{T_{k+1}^s}{\sum_{j=1}^{k+1}T_j^s}}+(1-\mu_{\alpha_{k+1}}^{s})^{\frac{T_{k+1}^s}{\sum_{j=1}^{k+1}T_j^s}}},\ \frac{\left(2\nu_{\alpha_{k+1}}^{s\,\frac{T_{k+1}^s}{\sum_{j=1}^{k+1}T_j^s}}\right)}{(2-\nu_{\alpha_{k+1}}^{s})^{\frac{T_{k+1}^s}{\sum_{j=1}^{k+1}T_j^s}}+\nu_{\alpha_{k+1}}^{s\,\frac{T_{k+1}^s}{\sum_{j=1}^{k+1}T_j^s}}}\right\},
$$

$$
=\left\{\frac{\prod_{j=1}^{k+1}(1+\mu_{\alpha_j}^{s})^{\frac{T_j^s}{\sum_{j=1}^{k+1}T_j^s}}-\prod_{j=1}^{k+1}(1-\mu_{\alpha_j}^{s})^{\frac{T_j^s}{\sum_{j=1}^{k+1}T_j^s}}}{\prod_{j=1}^{k+1}(1+\mu_{\alpha_j}^{s})^{\frac{T_j^s}{\sum_{j=1}^{k+1}T_j^s}}+\prod_{j=1}^{k+1}(1-\mu_{\alpha_k}^{s})^{\frac{T_j^s}{\sum_{j=1}^{k+1}T_j^s}}}\right.
$$

$$
\left.\frac{2\prod_{j=1}^{k+1}\nu_{\alpha_j}^{s\,\frac{T_j^s}{\sum_{j=1}^{k+1}T_j^s}}}{\prod_{j=1}^{k+1}(2-\nu_{\alpha_j}^{s})^{\frac{T_j^s}{\sum_{j=1}^{k+1}T_j^s}}+\prod_{j=1}^{k+1}\nu_{\alpha_j}^{s\,\frac{T_j^s}{\sum_{j=1}^{k+1}T_j^s}}}\right\} \tag{22}
$$

Hence the result (16) holds for $n = k + 1$. Therefore by mathematical induction method (16) holds for all n. Hence the theorem 1.

Theorem 2. Let $\alpha_j^s = (\mu_{\alpha_j}^s, \nu_{\alpha_j}^s),\ j = 1, 2, \ldots. n$ be the set of intuitionistic fuzzy numbers, $T_j^s = \sum_{k=1}^{j-1} s^*(\alpha_k^s),\ j = 2, 3, \ldots n,$ with $T_1 = 1$, where all IFNs are equal i.e. $\alpha_j^s = \alpha = (\mu_\alpha^s, \nu_\alpha^s)$ then IFEPWA $(\alpha_1^s, \alpha_2^s, \ldots, \alpha_n^s) = \alpha$ (23)

Proof: By definition, we have IFEPWA $(\alpha_1^s, \alpha_2^s, \ldots, \alpha_n^s)$

$$= \frac{T_1^s}{\sum_{j=1}^n T_j^s} e^{\alpha_1^s} \oplus \frac{T_2^s}{\sum_{j=1}^n T_j^s} e^{\alpha_2^s} \oplus \ldots \oplus \frac{T_n^s}{\sum_{j=1}^n T_j^s} e^{\alpha_n^s}$$

where $\alpha_1^s = \alpha_2^s = \alpha_3^s = \ldots = \alpha_n^s = \alpha$

= IFEPWA $(\alpha_1^s, \alpha_2^s, \ldots, \alpha^s)$

$$= \left\{ \frac{T_1^s}{\sum_{j=1}^n T_j^s}.e^{\alpha^s} \oplus \frac{T_2^s}{\sum_{j=1}^n T_j^s}.e^{\alpha^s} \oplus \ldots \oplus \frac{T_n^s}{\sum_{j=1}^n T_j^s}.e^{\alpha^s} \right\}$$

$$= \left\{ \frac{\prod_{j=1}^n (1+\mu_2^s)^{\frac{T_j^s}{\sum_{j=1}^n T_j^s}} - \prod_{j=1}^n (1-\mu_\alpha^s)^{\frac{T_j^s}{\sum_{j=1}^n T_j^s}}}{\prod_{j=1}^n (1+\mu_\alpha^s)^{\frac{T_j^s}{\sum_{j=1}^n T_j^s}} + \prod_{j=1}^n (1-\mu_\alpha^s)^{\frac{T_j^s}{\sum_{j=1}^n T_j^s}}}, \right.$$

$$\left. \frac{2\prod_{j=1}^n \nu_\alpha^{s\,\frac{T_j^s}{\sum_{j=1}^n T_j^s}}}{\prod_{j=1}^n (2-\nu_\alpha^s)\sum_{j=1}^{T_j} T_j^s + \prod_{j=1}^n \nu_\alpha^{s\,\frac{T_j^s}{\sum_{j=1}^n T_j^s}}} \right\}$$

$$= \left(\frac{1+\mu_\alpha^s - 1 + \mu_\alpha^s}{1+\mu_\alpha^s + 1 - \mu_\alpha^s}, \frac{2\nu_\alpha^s}{2-\nu_\alpha^s + \nu_\alpha^s} \right)$$

$$= (\mu_\alpha^s, \nu_\alpha^s) \tag{24}$$

Corollary 1. If $\alpha_j = (\mu_{\alpha_j}, \nu_{\alpha_j}),\ j = 1, 2, \ldots, n$ is collection of the largest IFNs i.e. $\alpha_j = \alpha^* = (1, 0),\ \forall\ j$ then

IFEPWA $(\alpha_1, \alpha_2, \ldots, \alpha_n) = IFEPWA(\alpha^*, \alpha^*, \ldots \alpha^*) = (1, 0)$ (25)

Theorem 3 (Monotonicity): Let $\alpha_j^s = (\mu_{\alpha_j}^s, \nu_{\alpha_j}^s)$ and

$\alpha_j'^s = (\mu_{\alpha_j'^s}, \nu_{\alpha_j'^s}),\ j = 1, 2, \ldots, n$ be two sets of IFNs,

$$T_j^s = \prod_{k=1}^{j-1} s^*(\alpha_k^s)$$

$$T_j'^s = \prod_{k=1}^{j=1} s^*(\alpha_k'^s),\ T_1^s = T_1'^s = 1,\ j = 2,3,....n,$$ Also let

$s^*(\alpha_k^s)$ and $s^*(\alpha_k'^s)$ be the series of $\alpha_k^s(\mu_{\alpha_k}^s, \nu_{\alpha_k}^s)$ and $\alpha_k'^s = (\mu_{\alpha_k}^s, \nu_{\alpha_k}^s)$ and

$\alpha_k'^s = (\mu_{\alpha_k}^s, \nu_{\alpha_k}^s)$, respectively .

If $\alpha_j^s \le \alpha_j'^s$ *i.e.* $\mu_{\alpha_j}^s \le \mu_{\alpha_j}'^s$ *and* $\nu_{\alpha_j}^s \ge \nu_{\alpha_j}'^s, \forall\ j$

Then $IFEPWA\ (\alpha_1^s, \alpha_2^s, ...,\alpha_n^s) \le IFEPWA\ (\alpha_1'^s, \alpha_2'^s,....,\alpha_n'^s)$ (26)

Proof: We know that

$$f(x) = \frac{1-x}{1+x},\quad \forall x \in [0, 1]$$

is a decreasing function of x. If

$$\mu_{\alpha_j}^s \le \mu_{\alpha_j'}^s \qquad \forall\ j \Rightarrow f(\mu_{\alpha_j'}^s) \le f(\mu_{\alpha_j}^s)$$

i.e. $\dfrac{1-\mu_{\alpha_j'}^s}{1+\mu_{\alpha_j'}^s} \le \dfrac{1-\mu_{\alpha_j}^s}{1+\mu_{\alpha_j}^s},\ \forall\ j$ (27)

Now let

$$w^s = \left(\frac{T_1^s}{\sum_{j=1}^{x} T_j^s}, \frac{T_2^s}{\sum_{j=1}^{x} T_j^s}, ..., \frac{T_n^s}{\sum_{j=1}^{x} T_j^s} \right)^T$$

and $w' = \left(\dfrac{T_1'^s}{\sum_{j=1}^{n} T_j'^s}, \dfrac{T_2'^s}{\sum_{j=1}^{n} T_j'^s}, ..., \dfrac{T_n'^s}{\sum_{j=1}^{x} T_j'^s} \right)$

be the prioritized weight vector of $\alpha_j^s = (\mu_{\alpha_j}^s, \nu_{\alpha_j}^s)$ and $\alpha_j'^s = (\mu_{\alpha_j}'^s, \nu_{\alpha_j}'^s)$, $j = 1, 2, ...,n$, such that

$\dfrac{T_j^s}{\sum_{j=1}^{n} T_j^s}, \dfrac{T_j'^s}{\sum_{j=1}^{n} T_j'^s} \in [0, 1]$ with the condition that

$$\sum_{j=1}^{n} \left(\frac{T_j'^s}{\sum_{j=1}^{n} T_j'^s} \right) = 1.$$

From (27), we have

$$\left(\frac{1-\mu_{a_j}'^s}{1+\mu_{\alpha_j}^s} \right) \frac{T_j^s}{\sum_{j=1}^{\alpha}} \le \left(\frac{1-\mu_{\alpha}^s}{1+\mu_{\alpha}^s} \right) \frac{T_j^s}{\sum_{j=1}^{n} T_j^s} \tag{28}$$

Thus

$$\sum_{j=1}^{n}\left(\frac{1-\mu^{s}_{\alpha'_j}}{1+\mu^{s}_{\alpha'_j}}\right)\frac{T'^{s}_{j}}{\sum_{j=1}^{n}T'^{s}_{j}} \leq \prod_{j=1}^{n}\left(\frac{1-\mu^{s}_{\alpha'_j}}{1+\mu^{s}_{\alpha'_j}}\right)\frac{T^{s}_{j}}{\sum_{j=1}^{n}T^{s}_{j}} \leq$$

$$\Leftrightarrow \quad \frac{2}{1+\prod_{j=1}^{n}\left(\frac{1-\mu'^{s}_{\alpha_j}}{1+\mu'^{s}_{\alpha_j}}\right)\frac{T^{s}_{j}}{\sum_{j=1}^{n}T^{s}_{j}}}-1$$

$$\leq \frac{2}{1+\prod_{j=1}^{n}\left(\frac{1-\mu^{s}_{\alpha_j}}{1+\mu^{s}_{\alpha'_j}}\right)\frac{T'^{s}_{j}}{\sum_{j=1}^{n}T'^{s}_{j}}}-1 \tag{29}$$

i.e. $$\frac{\prod_{j=1}^{n}\left(1+\mu^{s}_{\alpha}\frac{T^{s}_{j}}{\sum_{j=1}^{n}T^{s}_{j}}-\prod_{j=1}^{n}(1-\mu^{s}_{\alpha_j})\right)\frac{T^{s}_{j}}{\sum_{j=1}^{n}T^{s}_{j}}}{\prod_{j=1}^{n}(1+\mu^{s}_{\alpha_j})\frac{T^{s}_{j}}{\sum_{j=1}^{n}T^{s}_{j}}+\prod_{j=1}^{n}(1-\mu^{s}_{\alpha_j})\frac{T^{s}_{j}}{\sum_{j=1}^{n}T^{s}_{j}}}$$

$$\leq \frac{\prod_{j=1}^{n}(1+\mu^{s}_{\alpha'_j})\frac{T'^{s}_{j}}{\sum_{j=1}^{n}T'^{s}_{j}}-\prod_{j=1}^{n}(1-\mu^{s}_{\alpha'_j})\frac{T'^{s}_{j}}{\sum_{j=1}^{n}T'^{s}_{j}}}{\prod_{j=1}^{n}(1+\mu^{s}_{\alpha'_j})\frac{T'^{s}_{j}}{\sum_{j=1}^{n}T'^{s}_{j}}+\prod_{j=1}^{n}(1-\mu^{s}_{\alpha'_j})\frac{T'^{s}_{j}}{\sum_{j=1}^{n}T'^{s}_{j}}} \tag{30}$$

Now using the function

$g(y)=\dfrac{2-y}{y},\ \ y\in[0,1]$, a decreasing function of *y*, if $\nu^{s}_{\alpha_j}\geq \nu^{s}_{\alpha'_j},\ \ \forall\, j$, then

$g(\mu^{s}_{\alpha'_j})\geq g(\mu^{s}_{\alpha_j})$

i.e. $\dfrac{2-\nu^{s}_{\alpha'_j}}{\nu^{s}_{\alpha'_j}}\geq\dfrac{2-\nu^{s}_{\alpha_j}}{\nu^{s}_{\alpha_j}},\ \ \forall\, j$

Then

$$\left(\frac{2-\nu^{s}_{\alpha'_j}}{\nu^{s}_{\alpha'_j}}\right)\frac{T'^{s}}{\sum_{j=1}^{n}T'^{s}_{j}}\geq\left(\frac{2-\nu^{s}_{\alpha_j}}{\nu^{s}_{\alpha_j}}\right)\frac{T^{s}_{j}}{\sum_{j=1}^{n}T^{s}_{j}} \tag{31}$$

Thus $$\prod_{j=}^{n}\left(\frac{2-\nu^{s}_{\alpha'_j}}{\nu^{s}_{\alpha'_j}}\right)\frac{T'^{s}}{\sum_{j=1}^{n}T'^{s}_{j}}\geq\prod_{j=1}^{n}\left(\frac{2-\nu^{s}_{\alpha_j}}{\nu^{s}_{\alpha_j}}\right)\frac{T^{s}_{j}}{\sum_{j=1}^{n}T^{s}_{j}} \tag{32}$$

$$\Leftrightarrow \quad \frac{2}{\prod_{j=1}^{n}\left(\frac{2-\nu_{\alpha_j}^s}{\nu_{\alpha_j}^s}\right)^{\frac{T_j^s}{\sum_{j=1}^{n} T_j^s}}} \geq \frac{2}{\prod_{j=}^{n}\left(\frac{2-\nu_{\alpha'_j}^s}{\nu_{\alpha'_j}^s}\right)^{\frac{T'^s}{\sum_{j=1}^{n} T'^s_j}}} \tag{33}$$

i.e. $$\frac{2\prod_{j=1}^{n} \nu_{\alpha_j}^{s\,\frac{T_j^s}{\sum_{j=1}^{n} T_j^s}}}{\prod_{j=1}^{n} (2-\nu_{\alpha_j}^s)^{\frac{T_j^s}{\sum_{j=1}^{n} T_j^s}} + \prod_{j=}^{n} \nu_{\alpha_j}^{s\,\frac{T^s}{\sum_{j=1}^{n} T_j^s}}}$$

$$\geq \frac{2\prod_{j=1}^{n} \nu_{\alpha'_j}^{s\,\frac{T'^s_j}{\sum_{j=1}^{n} T'^s_j}}}{\prod_{j=1}^{n} (2-\nu_{\alpha'_j}^s)^{\frac{T'^s_j}{\sum_{j=1}^{n} T'^s_j}} + \prod_{j=}^{n} \nu_{\alpha'_j}^{s\,\frac{T'^s_j}{\sum_{j=1}^{n} T'^s_j}}} \tag{34}$$

Now (34) holds also even of $\nu_{\alpha}^s = \nu_{\alpha'}^s = 0, \forall\ j$ then by definition we obtain that

IFGEPWA $(\alpha_1^s, \alpha_2^s, ..., \alpha_n^s) \leq IFEPWA(\alpha'^s_1, \alpha'^s_2, ..., \alpha'^s_n)$

- **Relationship between Intuitionistic Fuzzy Einstein Prioritized Weighted Average (IFEPWA) operator and intuitionistic fuzzy prioritized weighted average (IFPWA) operator.**

Yu [10, 11, 12] proposed

IFPWA $(\alpha_1, \alpha_2, ..., \alpha_n)$

$$= \left(\frac{T_1}{\sum_{j=1}^{n} T_j}\alpha_1 \oplus \frac{T_2}{\sum_{j=1}^{n} T_j}\alpha_2 \oplus ...... \oplus \frac{T_n}{\sum_{j=1}^{n} T_n}\right)$$

$$= \left(1-\prod_{j=1}^{n} (1-\mu_{\alpha_j})^{\frac{T_1}{\sum_{j=1}^{n} T_j}}, \prod_{j=1}^{n} (\nu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{n} T_j}}\right) \tag{35}$$

Theorem : Let $\alpha_j^s = (\mu_{\alpha_j}^s, \nu_{\alpha_j}^s),\ j = 1, 2, ..., n$ be a set of intuitionistic fuzzy numbers, $T_j^s = \prod_{j=1}^{n-1} s^*(\alpha_k^s),\ j = 2, 3, ..., n\ T_1^s = 1$ and $s^*(\alpha_k^s)$ be the score of $\alpha_k^s = (\mu_{\alpha_k}^s, \nu_{\alpha_k}^s)$. Then

$$IFEPW(\alpha_1^s, \alpha_2^s, ..., \alpha_n^s) \leq IFPWA(\alpha_1^s, \alpha_2^s, ..., \alpha_n^s)$$

with equality iff $\alpha_1^s = \alpha_2^s = \alpha_3^s .... = d_n^s$ (36)

Proof: Using weighted AMGM inequality [15, 41], we have

$$\prod_{j=1}^{n}(1+\mu_{\alpha_j}^{s})^{\frac{T_j^s}{\sum_{j=1}^{n}T_j^s}}+\prod_{j=1}^{n}(1-\mu_{\alpha_j}^{s})^{\frac{T_j^s}{\sum_{j=1}^{n}T_j^s}}$$

$$\leq \sum_{j=1}^{n}\left(\frac{T_j^s}{\sum_{j=1}^{n}T_j^s}(1+\mu_{\alpha_j}^{s})+\sum_{j=1}^{n}\frac{T_j^s}{\sum_{j=1}^{n}T_j^s}(1-\mu_{\alpha_j}^{s})\right)=2 \tag{37}$$

Then

$$\frac{\prod_{j=1}^{n}(1+\mu_{\alpha_j}^{s})^{\frac{T_j^s}{\sum_{j=1}^{n}T_j^s}}-\prod_{j=1}^{n}(1-\mu_{\alpha_j}^{s})^{\frac{T_j^s}{\sum_{j=1}^{n}T_j^s}}}{\prod_{j=1}^{n}(1+\mu_{\alpha_j}^{s})^{\frac{T_j^s}{\sum_{j=1}^{n}T_j^s}}+\prod_{j=1}^{n}(1-\mu_{\alpha_j}^{s})^{\frac{T_j^s}{\sum_{j=1}^{n}T_j^s}}}$$

$$=1-\frac{2\prod_{j=1}^{n}(1-\mu_{\alpha_j}^{s})^{\frac{T_j^s}{\sum_{j=1}^{n}T_j^s}}}{\prod_{j=1}^{n}(1+\mu_{\alpha_j}^{s})^{\frac{T_j^s}{\sum_{j=1}^{n}T_j^s}}+\prod_{j=1}^{n}(1-\mu_{\alpha_j}^{s})^{\frac{T_j^s}{\sum_{j=1}^{n}T_j^s}}}$$

$$\leq 1-\prod_{j=1}^{n}(1-\mu_{\alpha_j}^{s})^{\frac{T_j}{\sum_{j=1}^{n}T_j^s}} \tag{38}$$

Equality holds iff $\mu_{\alpha_1}=\mu_{\alpha_2}=\mu_{\alpha_3}=....=\mu_{1_x}$

Since

$$\prod_{j=1}^{n}(2-\nu_{\alpha_j}^{s})^{\frac{T_j^s}{\sum_{j=1}^{n}T_j^s}}+\prod_{j=1}^{n}(\nu_{\alpha_j}^{s})^{\frac{T_j^s}{\sum_{j=1}^{n}T_j^s}}$$

$$\leq \sum_{j=1}^{n}\left(\frac{T_j^s}{\sum_{j=1}^{n}T_j^s}(2-\nu_{\alpha_j}^{s})+\frac{T_j^s}{\sum_{j=1}^{n}T_j^s}\nu_{\alpha_j}^{s}\right)$$

$$=2 \tag{39}$$

we have

$$\frac{2\prod_{j=1}^{n}(\nu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{n}T_j}}}{\prod_{j=1}^{n}(2-\nu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{n}T_j}}+\prod_{j=1}^{n}(\nu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{n}T_j}}}\geq\prod_{j=1}^{n}(\nu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{n}T_j}} \qquad (40)$$

when the equality holds iff

$\nu_{\alpha_1}=\nu_{\alpha_2}....\nu_{\alpha_n}$

Hence by definition, we have

$IFEPWA\,(\alpha_1^s,\alpha_2^s,....,\alpha_n^s)\leq IFPWA(\alpha_1,\alpha_2,....,\alpha_n)$

with equality iff $\alpha_1=\alpha_2=....=\alpha_n$

This proves the theorem.

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COMPARATIVE PROFIT ANALYSIS OF TWO RELIABILITY MODELS WITH VARYING DEMAND

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ABSTRACT :
The concerned paper analyses the comparison of a single unit system (Model 1) with scheduled maintenance and a two-unit cold standby system (Model 2) where both the units may become operative depending on the demand. The comparison of both systems is done by plotting different graphs between MTSFs (mean time to system failure), steady state availabilities and profit functions. Semi-Markov process and regenerative point technique has been used to analyze both the systems.
Keywords: ***Single unit system, Scheduled maintenance, Two–unit cold standby, semi-Markov process, Regenerative point technique.***

1. INTRODUCTION

No single model can be proved to be indispensible in every situation; neither can any one model perform at best in every situation. A particular model may perform better in one and may show a lower level utility in the other situation. Under this scenario, it becomes a critical obligation of the researcher to undertake a comparative study for the same. These studies are contributed by various researchers [1-9]. Maintenance has become the most important key to increase the availability and profit of an industrial system. However, many companies use the standby systems for improving the reliability. Thus, keeping the aspect of redundancy along with variation in demand in view, the author, in the present paper compared MTSFs, availabilities and profits of two different models using semi-Markov processes and regenerative point technique.

2. MATERIALS AND METHODS

d	Symbol for demand
n_1/ n_2	Symbol for production by one unit /two units
C_S	Unit is in cold standby state
Op/ O_{SM}	Operative unit / operative unit under scheduled maintenance
D	Unit is in down unit
F_r/ F_R	Failed unit under repair/Repair of failed unit continuing from previous state
F_w	Failed unit waiting for repair
λ	Failure rate of the operative unit
γ_1/γ_2	Rate of decrease/increase of demand so as to become < / ≥ production
γ_3	Rate of going from upstate to downstate
γ_4	Rate of change of state from down to up when there is no produce with the system and demand is there
γ_{11}	Rate of increase of demand when demand is atleast equal to the production by one unit and less than that by two units ($n_1 \le d < n_2$)

γ_{12} Rate of decrease of demand so as to become < production by one unit ($d < n_1$)

γ_{21} Rate of further increase of demand when demand is atleast equal to production by two units ($d \geq n_2$)

γ_{22} Rate of decrease of demand when demand is atleast equal to the production by one unit and less than that by two units ($n_1 \leq d < n_2$)

β Rate of requirement of scheduled maintenance

β_2 Rate of doing scheduled maintenance

p_1 Probability that during repair time demand ≥ production

p_2 Probability that during repair time demand < production

$p_{ij}, p_{ij}^{(k)}$ Probability of transition from a regenerative state i to a regenerative state j without visiting any other state in (0, t], visiting state k once in (0, t] respectively.

Bi Busy period of the repairman to repair of i^{th} model

DTi Expected down time of i^{th} model

SMi Busy period of the repairman for scheduled maintenance of ith model

Vi Expected number of visits of the repairman of i^{th} model

μ_i Mean sojourn time in regenerative state i before transiting to any other state of i^{th} Model

Mi Mean Time to system failure of i^{th} model

AP(1)/(2)i/ Availability when ($d< n_1$)/ ($d \geq n_1$) of i^{th} model

AD Availability when ($d \geq n_2$) of 2^{nd} Model

NPi Net profit of i^{th} model

C_0/ C_1/ C_R Revenue per unit up time when ($d < n_1$)/($n_1 \leq d < n_2$)/($d \geq n_2$)

C_2 Cost per unit time for engaging the repairman for repair

C_3/ C_{SM} Cost per unit time of the repairman/ for scheduled maintenance

C_4 Loss per unit time during the system remains down

g(t),G(t) p.d.f and c.d.f of repair time of unit

3. DESCRIPTION OF MODEL 1 AND MODEL 2:

Fig. 1 depicts the different states of the Model 1. (Refer [5]).

Profit = Difference of expected total revenue and expected total cost

$$\text{Profit (P1)} = (C_0 * AP(1)1 + C_1 * AP(2)2 - (C_2 * B1 + C_3 * V1 + C_4 * DT1 + C_{SM} * SM1)$$

$$M1 = \frac{\mu_0 + p_{01}\mu_1 + p_{01}\, p_{13}\mu_3}{p_{01}\, p_{14} + p_{02}}, \quad AP(1)1 = \frac{(1-p_{14}p_{41})\mu_0}{D_1}, \quad AP(2)1 = \frac{p_{01}\mu_1}{D_1}, p_{20} = 1,\ p_{30} = 1,\ p_{40} = p_1,\ p_{41} = p_2,$$

$$B1 = \frac{p_{02}(1-p_{14}p_{41})\mu_2 + p_{01}p_{14}\,\mu_4}{D_1}, \quad V1 = \frac{p_{02}(1-p_{14}p_{41}) + p_{01}p_{14}}{D_1}, \quad SM1 = \frac{p_{02}(1-p_{14}p_{41})p_{13}}{D_1},$$

$$DT1 = \frac{p_{01}\, p_{15}\,\mu_5}{D_1}, \quad D_1 = (1-p_{14}p_{41})\,\mu_0 + p_{01}\,\mu_1 + p_{02}\,(1-p_{14}p_{41})\,\mu_2 + p_{01}(1-p_{14}p_{41})\mu_3 + p_{01}p_{14}\,\mu_4 + p_{01}p_{15}\,\mu_5,$$

$$p_{01} = \frac{\gamma_1}{(\lambda+\gamma_1)},\ p_{02} = \frac{\lambda}{(\lambda+\gamma_1)},\ p_{10} = \frac{\gamma_2}{(\lambda+\gamma_2+\gamma_3)},\ p_{13} = \frac{\gamma_3}{(\lambda+\gamma_2+\gamma_3)},\ p_{14} = \frac{\lambda}{(\lambda+\gamma_2+\gamma_3)},$$

$$\mu_0 = \frac{1}{(\lambda+\gamma_1+\beta)},\ \mu_1 = \frac{1}{(\lambda+\gamma_2+\gamma_3)},\ \mu_2 = \int_0^\infty \overline{G}(t)dt = \mu_4,\ \mu_3 = \frac{1}{\beta_2}, \mu_5 = \frac{1}{\gamma_4}$$

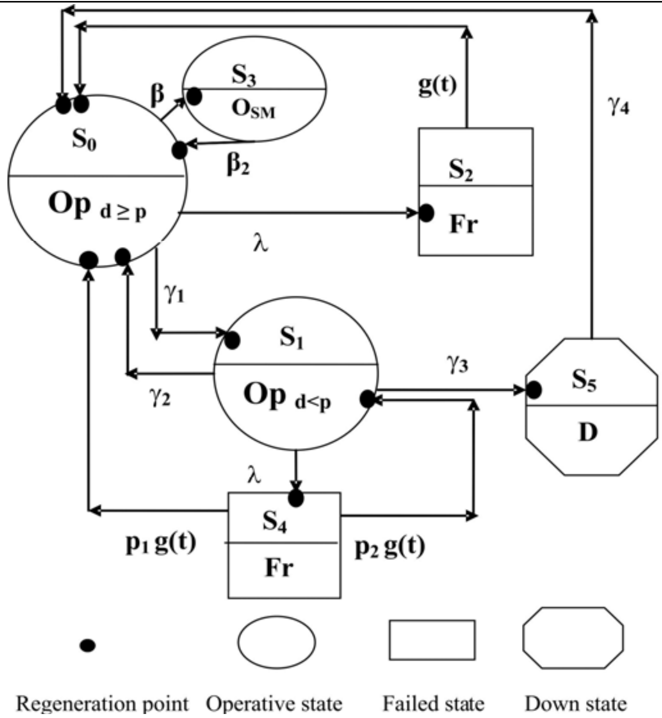

Fig. 1: State Transition Diagram of Model 1

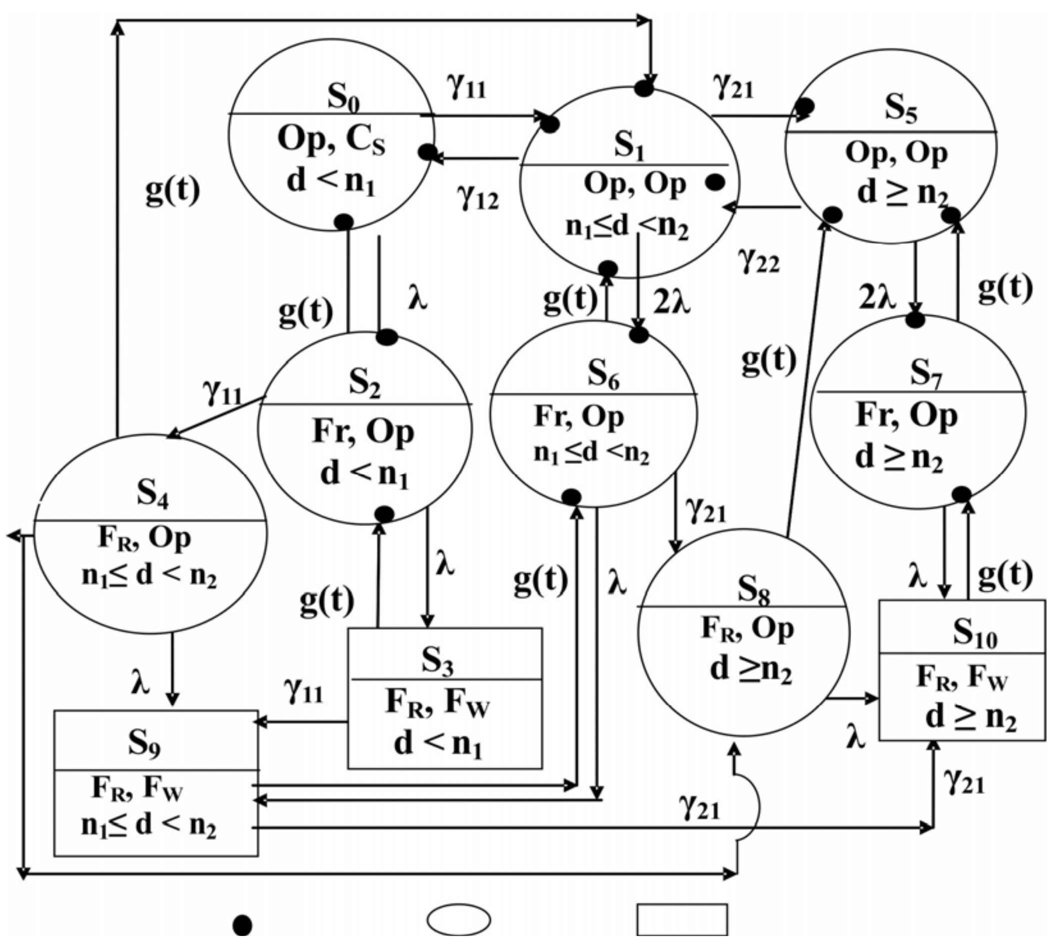

Fig.2: State Transition Diagram of Model 2

Fig. 2 shows the possible transitions of the Model 2. (Refer [6])

Profit $P = (C_0\, AP(1)2 + C_1\, AP(2)2 + C_R AD) - C_2 B2 - C_3 V2$

where MTSF=N/D, $AP(1)2=N_1/D_1$, $AP(2)2=N_2/D_1$, $AD= N_3/ D_1$,

$B_0 = \lim_{s\to 0}(sB_0{}^*(s)) = N_4/D_1$, $V_0 = \lim_{s\to 0}(sV_0{}^{**}(s)) = N_5/D_1$

$$
\begin{aligned}
N= {}& \mu_0[(1-p_{57}p_{75})(p_{16}(p_{69}+p_{6,10}^{8})+p_{10})+p_{7,10}(p_{15}p_{57}+p_{16}p_{57}p_{68}^{5})]+\mu_1[p_{01}(1-p_{57}p_{75})\\
&+p_{02}p_{21}^{4}(1-p_{57}p_{75})+p_{51}p_{02}p_{25}^{48}]+\mu_5[((1-p_{02}p_{20})(p_{15}+p_{16}p_{65}^{8})+p_{02}p_{10}p_{25}^{(4,8)})\\
&-(p_{23}+p_{24}^{9}+p_{2,10}^{(4,8)})p_{02}(p_{15}+p_{16}p_{65}^{8})+(p_{69}+p_{6,10}^{8})p_{02}p_{16}p_{25}^{48}]+\mu_7[p_{02}((1-p_{57}p_{75})\\
&(1-p_{16}p_{61})-p_{51}(p_{15}+p_{16}p_{65}^{8}))+p_{01}p_{16}(1-p_{57}p_{75})-p_{02}p_{16}(-p_{21}^{4}(1-p_{57}p_{75})-p_{51}p_{25}^{48})\\
&+p_{57}(p_{01}(p_{15}+p_{16}p_{65}^{8})+p_{02}(p_{25}^{48}+p_{21}^{4}p_{15}+p_{16}(p_{21}^{4}p_{65}^{8}-p_{61}p_{25}^{48})))]
\end{aligned}
$$

$$
\begin{aligned}
D= {}& (1-p_{57}p_{75})(1-p_{16}p_{61}-p_{01}p_{10}-p_{02}(p_{10}p_{21}^{4}+p_{20}(1-p_{16}p_{61})))-p_{51}((1-p_{02}p_{20})\\
&(p_{15}+p_{16}p_{65}^{8})+p_{02}p_{10}p_{25}^{48})
\end{aligned}
$$

$$
\begin{aligned}
N_1 = {}& (\mu_0(1-p_{22}^{3})+\mu_2 p_{02})[(1-p_{66}^{9})((1-p_{77}^{10}-p_{57}p_{75})-p_{15}p_{51}(1-p_{77}^{10}))+(1-p_{77}^{10})\\
&(-p_{16}(p_{51}p_{65}^{8}+p_{61}))-p_{75}p_{16}(p_{67}^{(8,10)}+p_{67}^{(9,10)})+p_{61}p_{57}]
\end{aligned}
$$

$$
\begin{aligned}
D_1= {}& -\mu_0(((1-p_{22}^{3})(1-p_{66}^{9}))(1-p_{77}^{10}-p_{57}p_{75})p_{10})+\mu_1[(1-p_{66}^{9})p_{15}((1-p_{77}^{10})((1-p_{22}^{3})\\
&-p_{02}p_{20})-k_1p_{02}p_{10}((1-p_{66}^{9})(1-p_{77}^{10}-p_{57}p_{75}))+\mu_5(((1-p_{77}^{10})((1-p_{22}^{3})-p_{02}p_{20})\\
&(p_{15}+p_{16}p_{65}^{8}))-p_{10}p_{02}((p_{26}^{(3,9)}+p_{26}^{(4,9)})((1-p_{77}^{10})p_{65}^{8}-p_{75}(p_{67}^{(8,10)}+p_{67}^{(9,10)})-\\
&(1-p_{66}^{9})(1-p_{77}^{10})p_{25}^{48}-p_{75}(p_{27}^{(3,9,10)}+p_{27}^{(4,8,10)}+p_{27}^{(4,9,10)}))))+k_1(((1-p_{22}^{3})-p_{02}p_{20})\\
&((1-p_{77}^{10})+p_{57}p_{75})-p_{10}p_{02}((1-p_{77}^{10})-p_{57}p_{75})(p_{26}^{(3,9)}+p_{26}^{(4,9)}))\\
&+k_1(p_{10}p_{57}((1-p_{66}^{9})(p_{01}(1-p_{22}^{3})+p_{02}p_{21}^{4})+p_{02}(p_{51}(p_{26}^{(3,9)}+p_{26}^{(4,9)})\\
&(p_{67}^{(8,10)}+p_{67}^{(9,10)})+(1-p_{66}^{9})(p_{27}^{(3,9,10)}+p_{27}^{(4,8,10)}+p_{27}^{(4,9,10)}))-p_{02}p_{61}p_{57}(p_{26}^{(3,9)}+p_{26}^{(4,9)}))\\
&+((1-p_{22}^{3})-p_{02}p_{20})((1-p_{66}^{9})p_{57}+p_{16}((p_{67}^{(8,10)}+p_{67}^{(9,10)})+p_{61}p_{57}))
\end{aligned}
$$

$$
\begin{aligned}
N_2 = {}& \mu_1[(1-p_{66}^{9})((1-p_{77}^{10})-p_{57}p_{75})(p_{01}(1-p_{22}^{3})+p_{02}p_{21}^{4})+p_{02}(1-p_{77}^{10})((p_{51}p_{65}^{8}+p_{61})(p_{26}^{(3,9)}+p_{26}^{(4,9)})\\
&+(1-p_{66}^{9})p_{25}^{48}p_{51})-p_{51}p_{75}((p_{67}^{(8,10)}+p_{67}^{(9,10)})(p_{26}^{(3,9)}+p_{26}^{(4,9)})-(1-p_{66}^{9})(p_{27}^{(3,9,10)}+p_{27}^{(4,8,10)}+p_{27}^{(4,9,10)}))\\
&+p_{61}p_{57}p_{75}(p_{26}^{(3,9)}+p_{26}^{(4,9)})]+\mu_7[p_{01}(1-p_{22}^{3})(p_{16}(1-p_{77}^{10})-p_{57}p_{75})-p_{02}(p_{16}((-p_{21}^{4}(1-p_{77}^{10})\\
&-p_{57}p_{75})+p_{51}(p_{25}^{48}(1-p_{77}^{10})-p_{75}(p_{27}^{(3,9,10)}+p_{27}^{(4,8,10)}+p_{27}^{(4,9,10)}))))+p_{02}(p_{26}^{(3,9)}+p_{26}^{(4,9)})((1-p_{77}^{10})\\
&(1-p_{15}p_{51})-p_{57}p_{75})]
\end{aligned}
$$

$$
\begin{aligned}
N_3 \; {}& = p_{01}(1-p_{22}^{3})(p_{15}(1-p_{66}^{9})+p_{16}p_{65}^{8})(-\mu_5p_{16}p_{75}(p_{67}^{(8,10)}+p_{67}^{(9,10)})+p_{57}\mu_7)\\
&-p_{02}((-p_{25}^{(4,8)}(1-p_{66}^{9})-(p_{65}^{8}+p_{15}p_{61})(p_{26}^{(3,9)}+p_{26}^{(4,9)})-p_{15}p_{21}^{4}(1-p_{66}^{9}))\\
&-p_{16}(p_{21}^{4}p_{65}^{8}-p_{61}p_{25}^{(4,8)}))((1-p_{77}^{10})\mu_5+p_{57}\mu_7)-\mu_5p_{02}\;[-(p_{27}^{(3,9,10)}+p_{27}^{(4,8,10)}+p_{27}^{(4,9,10)})\\
&(p_{75}(1-p_{66}^{9})-p_{16}p_{61}p_{75})-p_{75}(p_{67}^{(8,10)}+p_{67}^{(9,10)})((p_{26}^{(3,9)}+p_{26}^{(4,9)})+p_{16}p_{21}^{4})]-\mu_7[-p_{02}\\
&(p_{27}^{(3,9,10)}+p_{27}^{(4,8,10)}+p_{27}^{(4,9,10)})((1-p_{66}^{9})(1-p_{15}p_{51})-p_{16}(p_{51}p_{65}^{8}+p_{61}))-(p_{67}^{(8,10)}+p_{67}^{(9,10)})\\
&(p_{01}p_{16}(1-p_{22}^{3})-p_{02}((1-p_{15}p_{51})(p_{26}^{(3,9)}+p_{26}^{(4,9)})+p_{16}(p_{21}^{4}+p_{51}p_{25}^{(4,8)})))]
\end{aligned}
$$

$$
\begin{aligned}
N_4 = \;& k_1[p_{02}((1 - p_{66}^{9})((1- p_{77}^{10} - p_{57}p_{75}) - p_{15}p_{51}(1- p_{77}^{10})) + (1- p_{77}^{10}) \\
& (-p_{16}(p_{51}p_{65}^{8} + p_{61})) - p_{75}p_{16}((p_{67}^{(8,10)} + p_{67}^{(9,10)}) + p_{61}p_{57})) + (p_{01}(1 - p_{22}^{3})(p_{16}(1- p_{77}^{10}) \\
& - p_{57}p_{75}) - p_{02}(p_{16}((-p_{21}^{4}(1- p_{77}^{10}) - p_{57}p_{75} + p_{51}(p_{25}^{48}(1- p_{77}^{10}) - p_{75}(p_{27}^{(3,9,10)} + p_{27}^{(4,8,10)} \\
& + p_{27}^{(4,9,10)})))) + p_{02}(p_{26}^{(3,9)} + p_{26}^{(4,9)})((1- p_{77}^{10})(1- p_{15}p_{51}) - p_{57}p_{75})) - p_{01}(1- p_{22}^{3}) \\
& (p_{15}(1- p_{66}^{9}) + p_{16}p_{65}^{8})p_{57} - p_{02}((-p_{25}^{(4,8)}(1- p_{66}^{9}) - (p_{65}^{8} + p_{15}p_{61})(p_{26}^{(3,9)} + p_{26}^{(4,9)}) - \\
& p_{15}p_{21}^{4}(1 - p_{66}^{9})) - p_{16}(p_{21}^{4}p_{65}^{8} - p_{61}p_{25}^{(4,8)}))p_{57} - (-p_{02}(p_{27}^{(3,9,10)} + p_{27}^{(4,8,10)} + p_{27}^{(4,9,10)}) \\
& ((1 - p_{66}^{9})(1- p_{15}p_{51}) - p_{16}(p_{51}p_{65}^{8} + p_{61})) - (p_{67}^{(8,10)} + p_{67}^{(9,10)})(p_{01}p_{16}(1 - p_{22}^{3}) - \\
& p_{02}(1- p_{15}p_{51})(p_{26}^{(3,9)} + p_{26}^{(4,9)}) + p_{16}(p_{21}^{4} + p_{51}p_{25}^{(4,8)})))))]
\end{aligned}
$$

$$
\begin{aligned}
N_5 = \;& p_{02}(1 - p_{22}^{3})[(1 - p_{66}^{9})((1- p_{77}^{10} - p_{57}p_{75}) - p_{15}p_{51}(1- p_{77}^{10})) + (1- p_{77}^{10})(-p_{16}(p_{51}p_{65}^{8} + p_{61})) \\
& - p_{75}p_{16}((p_{67}^{(8,10)} + p_{67}^{(9,10)}) + p_{61}p_{57})] - p_{16}[(1 - p_{66}^{9})((1- p_{77}^{10}) - p_{57}p_{75})(p_{01}(1 - p_{22}^{3}) + p_{02}p_{21}^{4}) \\
& + p_{02}(1- p_{77}^{10})((p_{51}p_{65}^{8} + p_{61})(p_{26}^{(3,9)} + p_{26}^{(4,9)}) + (1 - p_{66}^{9})p_{25}^{48}p_{51}) - p_{51}p_{75}((p_{67}^{(8,10)} + p_{67}^{(9,10)}) \\
& (p_{26}^{(3,9)} + p_{26}^{(4,9)}) - (1 - p_{66}^{9})(p_{27}^{(3,9,10)} + p_{27}^{(4,8,10)} + p_{27}^{(4,9,10)})) + p_{61}p_{57}p_{75}(p_{26}^{(3,9)} + p_{26}^{(4,9)})] \\
& - p_{57}p_{01}(1- p_{22}^{3})(p_{15}(1- p_{66}^{9}) + p_{16}p_{65}^{8})(-p_{16}p_{75}(p_{67}^{(8,10)} + p_{67}^{(9,10)})) - p_{02}((-p_{25}^{(4,8)}(1- p_{66}^{9}) \\
& - (p_{65}^{8} + p_{15}p_{61})(p_{26}^{(3,9)} + p_{26}^{(4,9)}) - p_{15}p_{21}^{4}(1 - p_{66}^{9})) - p_{16}(p_{21}^{4}p_{65}^{8} - p_{61}p_{25}^{(4,8)}))(1 - p_{77}^{10}) \\
& - p_{02}(-(p_{27}^{(3,9,10)} + p_{27}^{(4,8,10)} + p_{27}^{(4,9,10)})(p_{75}(1 - p_{66}^{9}) - p_{16}p_{61}p_{75}) - p_{75}(p_{67}^{(8,10)} + p_{67}^{(9,10)}) \\
& ((p_{26}^{(3,9)} + p_{26}^{(4,9)}) + p_{16}p_{21}^{4})]
\end{aligned}
$$

The values of probabilities p_{ij} are given by

$$p_{01} = \frac{\gamma_{11}}{(\lambda + \gamma_{11})} \qquad p_{02} = \frac{\lambda}{(\lambda + \gamma_{11})} \qquad p_{10} = \frac{\gamma_{12}}{(2\lambda + \gamma_{12} + \gamma_{21})}$$

$$p_{15} = \frac{\gamma_{21}}{(2\lambda + \gamma_{12} + \gamma_{21})} \qquad p_{16} = \frac{2\lambda}{(2\lambda + \gamma_{12} + \gamma_{21})} \qquad p_{20} = g^*(\lambda + \gamma_{11})$$

$$p_{21}^{4} = \frac{\gamma_{11}}{(\gamma_{21} - \gamma_{11})}(g^*(\lambda+\gamma_{11}) - g^*(\lambda+\gamma_{21})) \qquad p_{22}^{3} = (-g^*(\lambda+\gamma_{11}) + g^*(\gamma_{11}))$$

$$p_{23} = \frac{\lambda}{(\lambda+\gamma_{11})}(1 - g^*(\lambda+\gamma_{11})) \qquad p_{2,5}^{(4,8)} = \gamma_{11}\gamma_{21}\left(\frac{g^*(\lambda+\gamma_{11})}{(\gamma_{11}-\gamma_{21})\gamma_{11}} - \frac{g^*(\lambda+\gamma_{21})}{(\gamma_{11}-\gamma_{21})\gamma_{21}} + \frac{g^*(\lambda)}{(\gamma_{11}\gamma_{21})}\right)$$

$$p_{2,6}^{(3,9)} = \lambda\gamma_{11}\left(\frac{g^*(\lambda+\gamma_{11})}{\lambda(\lambda+\gamma_{11}-\gamma_{21})} - \frac{g^*(\gamma_{21})}{(\lambda+\gamma_{11}-\gamma_{21})(\gamma_{21}-\gamma_{11})} + \frac{g^*(\gamma_{11})}{\lambda(\gamma_{21}-\gamma_{11})}\right)$$

$$p_{2,6}^{(4,9)} = \lambda\gamma_{11}\left(\frac{-g^*(\lambda+\gamma_{11})}{(\lambda+\gamma_{11}-\gamma_{21})(\gamma_{21}-\gamma_{11})} + \frac{g^*(\lambda+\gamma_{21})}{\lambda(\gamma_{21}-\gamma_{11})} + \frac{g^*(\gamma_{21})}{\lambda(\lambda+\gamma_{11}-\gamma_{21})}\right)$$

$$p_{2,7}^{(3,9,10)} = \lambda\gamma_{11}\gamma_{21}\left(\frac{-g^*(\lambda+\gamma_{11})}{\lambda(\lambda+\gamma_{11})(\lambda+\gamma_{11}-\gamma_{21})} - \frac{g^*(\gamma_{11})}{\lambda\gamma_{11}(\gamma_{21}-\gamma_{11})} + \frac{g^*(\gamma_{21})}{\gamma_{21}(\gamma_{21}-\gamma_{11})(\lambda+\gamma_{11}-\gamma_{21})} + \frac{1}{\gamma_{11}\gamma_{21}(\lambda+\gamma_{11})}\right)$$

$$p_{2,7}^{(4,8,10)} = \lambda\gamma_{11}\gamma_{21}\left(\frac{g^*(\lambda+\gamma_{11})}{\gamma_{11}(\lambda+\gamma_{11})(\gamma_{21}-\gamma_{11})} - \frac{g^*(\lambda+\gamma_{21})}{\gamma_{21}(\lambda+\gamma_{21})(\gamma_{21}-\gamma_{11})} - \frac{g^*(\lambda)}{\lambda\gamma_{11}\gamma_{21}} + \frac{1}{\lambda(\lambda+\gamma_{11})(\lambda+\gamma_{21})}\right)$$

$$p_{2,7}^{(4,9,10)} = \lambda\gamma_{11}\gamma_{21}\left(\begin{array}{l}\frac{g^*(\lambda+\gamma_{11})}{(\lambda+\gamma_{11})(\lambda+\gamma_{11}-\gamma_{21})(\gamma_{21}-\gamma_{11})} - \frac{g^*(\lambda+\gamma_{21})}{\lambda(\lambda+\gamma_{21})(\gamma_{21}-\gamma_{11})} - \frac{g^*(\gamma_{21})}{\lambda\gamma_{21}(\lambda+\gamma_{11}-\gamma_{21})}\\ + \frac{1}{\gamma_{21}(\lambda+\gamma_{11})(\lambda+\gamma_{21})}\end{array}\right)$$

$$p_{29}^{4} = \frac{\lambda\gamma_{11}}{(\gamma_{21}-\gamma_{11})}\left(\frac{(1-g^*(\lambda+\gamma_{11}))}{(\lambda+\gamma_{11})} - \frac{(1-g^*(\lambda+\gamma_{21}))}{(\lambda+\gamma_{21})}\right)$$

$$p_{2,10}^{(4,8)} = \lambda\gamma_{11}\gamma_{21}\left(\frac{(1-g^*(\lambda+\gamma_{11}))}{\gamma_{11}(\lambda+\gamma_{11})(\gamma_{11}-\gamma_{21})} - \frac{(1-g^*(\lambda+\gamma_{21}))}{\gamma_{21}(\lambda+\gamma_{21})(\gamma_{11}-\gamma_{21})} + \frac{(1-g^*(\lambda))}{(\lambda\gamma_{11}\gamma_{21})}\right)$$

$$p_{51} = \frac{\gamma_{22}}{(2\lambda+\gamma_{22})} \qquad p_{57} = \frac{2\lambda}{(2\lambda+\gamma_{22})} \qquad p_{61} = g^*(\lambda+\gamma_{21})$$

$$p_{65}^{8} = g^*(\lambda+\gamma_{21}) + g^*(\lambda) \qquad p_{66}^{9} = -g^*(\lambda+\gamma_{21}) + g^*(\gamma_{21})$$

$$p_{67}^{(8,10)} = \lambda\gamma_{21}\left(\frac{g^*(\lambda+\gamma_{21})}{\gamma_{21}(\lambda+\gamma_{21})} - \frac{g^*(\lambda)}{(\lambda\gamma_{21})} + \frac{1}{\lambda(\lambda+\gamma_{21})}\right)$$

$$p_{67}^{(9,10)} = \lambda\gamma_{21}\left(\frac{g^*(\lambda+\gamma_{21})}{\lambda(\lambda+\gamma_{21})} - \frac{g^*(\gamma_{21})}{(\lambda\gamma_{21})} + \frac{1}{\gamma_{21}(\lambda+\gamma_{21})}\right)$$

$$p_{69} = \frac{\lambda}{(\lambda+\gamma_{21})}(1-g^*(\lambda+\gamma_{21})) \qquad p_{6,10}^{8} = \lambda\gamma_{21}\left(-\frac{(1-g^*(\lambda+\gamma_{21}))}{\gamma_{21}(\lambda+\gamma_{21})} + \frac{(1-g^*(\lambda))}{(\lambda\gamma_{21})}\right)$$

$$p_{6,10}^{8} = \lambda\gamma_{21}\left(-\frac{(1-g^*(\lambda+\gamma_{21}))}{\gamma_{21}(\lambda+\gamma_{21})} + \frac{(1-g^*(\lambda))}{(\lambda\gamma_{21})}\right)$$

$$p_{75} = g^*(\lambda) \qquad p_{77}^{10} = p_{7,10} = 1-g^*(\lambda)$$

$$\mu_0 = \frac{1}{(\lambda+\gamma_{11})} \quad \mu_1 = \frac{1}{(2\lambda+\gamma_{12}+\gamma_{21})} \quad \mu_2 = \frac{1}{(\lambda+\gamma_{11})}(1-g^*(\lambda+\gamma_{11}))$$

$$\mu_5 = \frac{1}{(2\lambda+\gamma_{22})} \quad \mu_6 = \frac{1}{(\lambda+\gamma_{21})}(1-g^*(\lambda+\gamma_{21})) \quad \mu_7 == \frac{1}{\lambda}(1-g^*(\lambda))$$

5. RESULTS AND DISCUSSION:

Fig. 3 shows the behavior of mean time to system failures (**M_1, M_2**) with respect to β_2. **Fig. 4** and **Fig. 5** shows the behavior of the availabilities for Model 1 and Model 2 when demand < production and demand ≥ production by one unit with respect to β_2. The values of the other parameters are γ_1=0.07/hr, γ_{21}=0.4213/hr, α=0.02/hr, γ_{22}=0.353/hr, p_1=0.665/hr, γ_{11}=0.235/hr, γ_2=0.235/hr, p_2=0.335/hr, γ_3=0.353/hr, γ_{12}=0.07/hr, γ_4=.4213/hr, β=0.0001/hr.

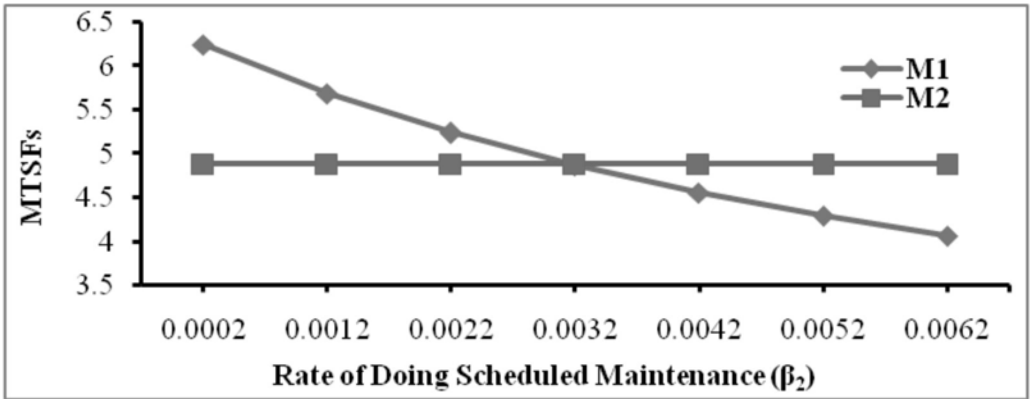

Fig. 3: Mean Time to System Failures (M1, M2) versus β_2

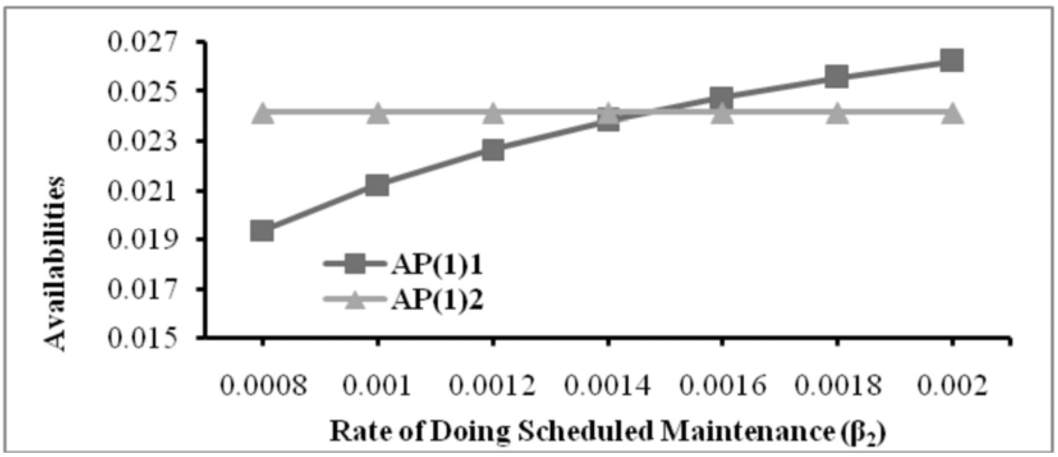

Fig. 4: (AP (1)1, AP (1)2) when ($d < n_1$) versus β_2

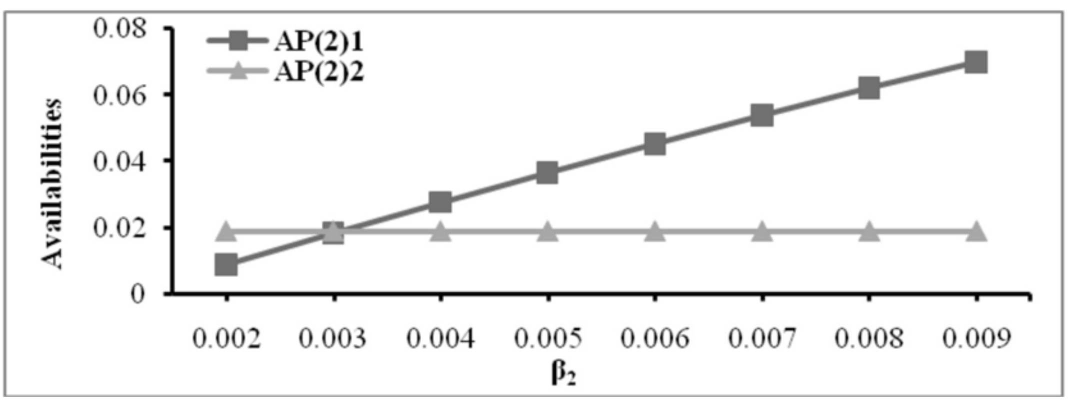

Fig. 5: (AP(2)1, AP(2)2) when ($d \geq n_1$) versus β_2

Profits (NP1, NP6) versus β have been shown in **Fig. 6.** The other parameters have values α=0.05/hr, γ_{22}=0.353/hr, p_2=0.335/hr, C_3= INR 100, γ_1=0.07/hr, γ_{21}=0.4213/hr, C_4=INR 2000, C_R=INR 800, γ_3=0.353/hr, L=INR 295, C_2=INR 200, γ_{11}=0.235/hr, C_1=INR 1000, p_1=0.665/hr, γ_4=.4213/hr, ICA=INR 500, γ_{12}=0.07/hr, γ_2=0.235/hr, β_2=0.002/hr, C_0=INR 7000, C_{SM}=INR 400.

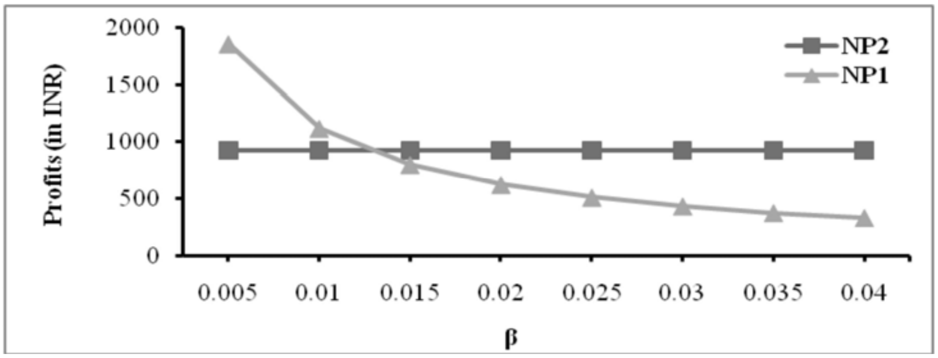

Fig. 6: Profits (NP1, NP2) versus β

Profits (NP2-NP1) versus C_0 for different values of C_3 have been shown in **Fig. 7**. The values of the other parameters are γ_{22} =0.353/hr, α=0.05/hr, C_4=INR 2000, γ_{21}=0.4213/hr, ICA=INR 500, β=0.0001/hr, L=INR 295, γ_1=0.07/hr, C_{SM}=INR 400, C_1=INR 1000, γ_2=0.235/hr, C_2=INR 200, γ_3=0.353/hr, C_3= INR 100, γ_4=.4213/hr, p_2=0.335/hr, γ_{12}=0.07/hr, γ_{11}=0.235/hr, β_2=0.002/hr, C_R=INR 800.

Profits (NP1, NP2) versus C_1 have been shown in **Fig. 8**. The values of the other parameters are L=INR 295, α=0.05/hr, γ_1=0.07/hr, ICA=INR 140, γ_2=0.235/hr, γ_3=0.353/hr, γ_4=.4213/hr, C_2=INR 200, p_2=0.335/hr, γ_{22} =0.353/hr, C_0=INR 7000, β_2=0.002/hr, C_{SM}=INR 400, γ_{12}=0.07/hr, β=0.0001/hr, C_3= INR 50000, γ_{11}=0.235/hr, C_4=INR 66000, γ_{21}=0.4213/hr, C_R=INR 800.

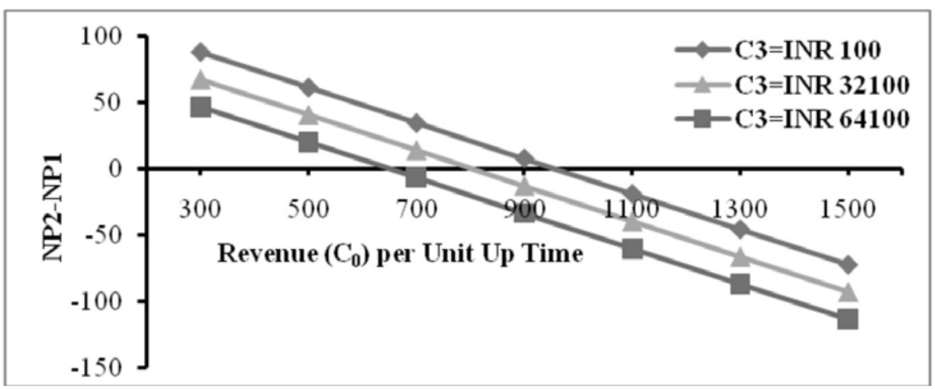

Fig. 7: Profits (NP2-NP1) versus C_0 for Different Values of Cost (C_3)

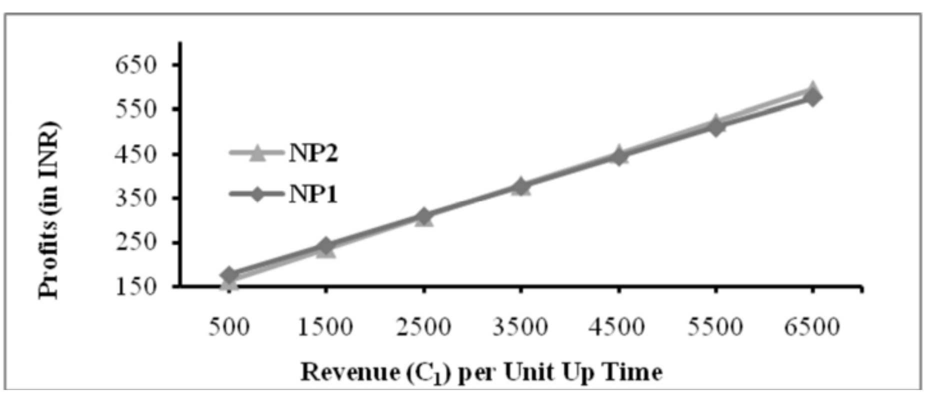

Fig. 8: Profits (NP1, NP2) versus C_1

Profits (NP1, NP2) versus C_4 have been shown in **Fig. 9**. The values of the other parameters are β=0.0001/hr, α=0.05/hr, p_1=0.665/hr, γ_{11}=0.235/hr, L=INR 295, γ_{22} =0.353/hr, C_{SM}=INR 400, ICA=INR 500, C_3= INR 100,

γ_4=.4213/hr, β_2=0.002/hr, C_R=INR 800, γ_1=0.07/hr, C_2=INR 200, γ_2=0.235/hr, C_0=INR 7000, γ_3=0.353/hr, C_1=INR 1000, γ_{12}=0.07/hr, γ_{21}=0.4213/hr.

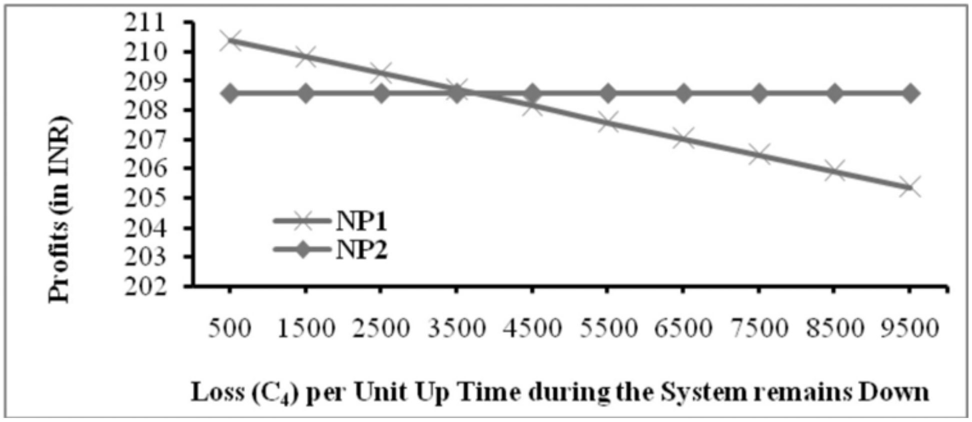

Fig. 9: Profits (NP1, NP2) versus C_4

Profits (NP1, NP2) versus ICA have been shown in **Fig. 10.** The values of other parameters are γ_2=0.235/hr, C_0=INR 7000, γ_{22} =0.353/hr, C_3= INR 100, α=0.05/hr, C_1=INR 1000, β_2=0.002/hr, C_R=INR 800, γ_1=0.07/hr, C_{SM}=INR 400, β=0.0001/hr, γ_3=0.353/hr, C_4=INR 2000, γ_4=.4213/hr, γ_{11}=0.235/hr, L=INR 295, γ_{12}=0.07/hr, C_2=INR 200, γ_{21}=0.4213/hr, p_2=0.335/hr.

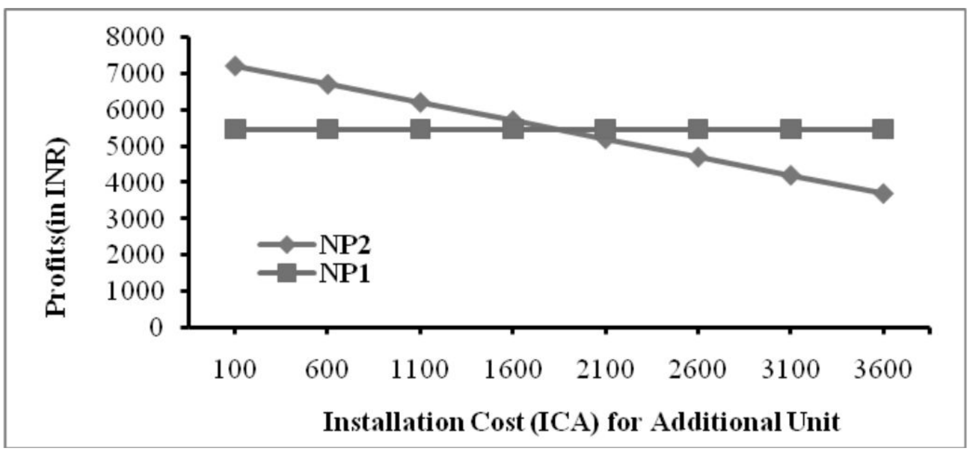

Fig. 10: Profits (NP1, NP2) versus ICA

5. CONCLUSIONS:

The focus of the comparative study is to explore and elaborate two different reliability models on the basis of MTSFs, availabilities and profits incurred. The comparison considers the particular cases considering exponential distributions. Cut-off points are obtained to determine effectiveness of each model. However, the users of such systems, while using the author's models, may select or take that distribution which fits best to the data on failures, repairs etc. available to them.

Table 1: Comparative Study between M_1 and M_2

Comparison with respect to		**Which Model is more effective (according to varying situations)**		
		M_1 is more effective if	**M_2 is more effective if**	**Both the Models are equally efffective**
MTSF		$\beta_2 < 0.003192$	$\beta_2 > 0.003192$	$\beta_2 = 0.003192$
AOP(2)		$\beta_2 > 0.00294$	$\beta_2 < 0.00294$	$\beta_2 = 0.00294$
AOP(1)		$\beta_2 > 0.00137$	$\beta_2 < 0.00137$	$\beta_2 = 0.00137$
Profit		$\beta < 0.001392$	$\beta > 0.001392$	$\beta = 0.001392$
Profit	C_3= INR 100	$C_0 > 969.155$	$C_0 < 969.155$	$C_0 = 969.155$
	C_3=INR 1100	$C_0 > 845.291$	$C_0 < 845.291$	$C_0 = 845.291$
	C_3=INR 64100	$C_0 > 680.175$	$C_0 < 680.175$	$C_0 = 680.175$
Profit		$C_1 < 2875.112$	$C_1 > 2875.112$	$C_1 = 2875.112$
Profit		$C_4 < 1012.193$	$C_4 > 1012.193$	$C_4 = 1012.193$
Profit		ICA>1852.431	ICA<1852.431	ICA = 1852.431

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ECONOMIC ANALYSIS OF A WATER PROCESS SYSTEM WITH PROVISION OF NON-SWITCHING AND ONLINE REPAIRS ON MINOR FAULTS

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ABSTRACT :
The paper investigates a stochastic model developed for an industrial process water system on the basis of observation taken from Panipat Thermal Power Plant, Panipat. The system comprises of several redundant and non-redundant subsystems wherein the redundant subsystems are of two types, namely 1-out-of-2 (Type-I) and 2-out-of-3 (Type-II). The system goes to complete failure due to a fault in non-redundant subsystem whereas it goes to partial failures due to a minor fault as to avoid the generation losses switching of the redundant subsystems is not carried out. However in case of occurrence of major faults in a redundant subsystem, switching of the subsystem is automatic and instantaneous. Further on partial failure, the repairman first inspects the system to judge which subsystem has the minor fault and accordingly carries out the online repair of the subsystems. For the model, various measures of system effectiveness are obtained using Markov process and regenerative point technique. Profit incurred to the system is also computed. The reliability and cost analyses of the system are presented through graphs for a particular case.

Keywords: ***Water process system, mean time to system failure, expected uptime with full / reduced capacity, profit, Markov process and regenerative point technique.***

INTRODUCTION

Water is used as a raw material, solvent, coolant, transport agents and energy source in various industrial productions and hence several industries are reliant on water. The water process systems have important role in the various industries like wind mills, power plants, milk plant, beverages and food industry etc. In fact in these industries, reliability and cost of water process systems play a very crucial role. Such industries can no longer ignore water management issues if they have to grow and compete globally in the market. In fact industrial water management is closely related to economy of a country. However water use and its management in the industries is a double edged sword as on one hand local water resources have been affected by the immense pressure it puts and on the other hand discharged water pollutes the local environment. As a result the researchers in different areas are taking close look at the alternative energy production or conversion devices besides developing new techniques for better utilization of the available water resources.

In the light of the above for analyzing a water process system, operation of a water process system at the Panipat Thermal Power Plant, Panipat was observed and real data regarding various faults, maintenances, inspections, repairs etc. of the system was collected. It was observed that the water process system has important role in the thermal power plant due to the dependency of whole process on circulation of water in different modes through various subsystems. Some of these subsystems are redundant like raw water pump, condensate exhaust pump and boiler feed pump etc. and others are non-redundant subsystems like gland steam cooler, low pressure heater, economizer and boiler drum etc. The water process system consisting two identical units comprises of several redundant and non-redundant subsystems wherein the redundant subsystems are of two types, namely 1-out-of-2 (Type-I) and 2-out-of-3 (Type-II). These subsystems have different type of faults, some of these are minor faults such as vibration in motor of raw water pump, tripping in service water pump etc. and others are major faults such as casing leakage in main boiler feed pump (BFP), cartridge damaged in BFP etc. On occurrence of a minor fault, the system goes to partial failure whereas in case of a major fault the system goes to complete failure. Further major fault in non-redundant subsystem leads to a complete failure of the system whereas on occurrence of major fault in redundant subsystem the system remains operative due to automatic switching. Moreover, on partial failure of the system, the repairman first inspects the unit which subsystem has the minor fault and accordingly carries out the online repair of the subsystem, i.e. without halting the operation of the system, to avoid losses (including loss of power generation) etc.

Several researchers in the field of reliability modeling including Garg and Kumar (1977), Goel et al (1986), Gopalan, and Murlidhar (1991), Li et al (1998), Mokaddis et al (1997), Kumar et al (2001), Taneja et al (2004), Levitin (2010), Kumar et al. (2010), Kumar and Bhatia (2011), Kumar and Kumar (2012), Kumar and Rani (2013), Kumar and Kapoor (2013) etc. discussed the reliability and cost-benefit analyses of different one and two-unit systems considering several aspects of the systems such as different types of faults, online/offline maintenances, repairs, inspections, rest/halt of system, degradation etc. Considering the above practical situations in the industry, none of the researcher analysed a water process system. The present paper is an attempt in this direction.

The paper deals with the economic analysis for the industrial process water system through a stochastic model working in a thermal power plant comprises of several redundant and non-redundant subsystems. The redundant subsystems are considered to be of two types, namely 1-out-of-2 (Type-I) and 2-out-of-3 (Type-II). The system goes to complete failure due to a fault in non-redundant subsystem whereas it goes to partial failures due to a minor fault as to avoid the generation losses switching of the redundant subsystems is not carried out. However in case of occurrence of major faults in a redundant subsystem, switching of the subsystem is automatic and instantaneous. Further on partial failure, the repairman first inspects the system to judge which subsystem has the minor fault and accordingly carries out the online repair of the subsystems. Other assumptions taken in the model are:

1. The system is initially operative and the various faults are self- announcing.
2. There is single repair facility that reaches the system in negligible time.
3. The priority for repair is given to the non-redundant subsystem.
4. The times to repair varies with the subsystems.
5. After each repair the system is as good as new.
6. The failure time distributions are taken exponential while other time distributions are considered general.
7. All the random variables are mutually independent.

For the model, various measures of system effectiveness are obtained using Markov process and regenerative point technique. Profit incurred to the system is also computed. The reliability and cost analyses of the system are presented through graphs for a particular case.

STATES OF THE SYSTEM

O	:	Operative system.
O_i/F_i	:	Operative / failed system under inspection.
O_{NRDr}/F_{NRDr}	:	Operative / failed non-redundant subsystem under repair.
O_{RDi}	:	Operative redundant subsystem under inspection.
$O_{RD\text{-}I}/O_{RD\text{-}II}$	:	Operative redundant subsystem of Type-I/II.
$O_{RD\text{-}Ir}/O_{RD\text{-}IIr}$	:	Operative redundant subsystem of Type-I/II under repair.
$F_{RD\text{-}Ir}/F_{RD\text{-}IIr}$	:	Failed redundant subsystem of Type-I/II under repair.
$F_{RD\text{-}IR}/F_{RD\text{-}IIR}$	:	Failed redundant subsystem of Type-I/II under repair from the previous state.
F_r	:	Failed system under repair.

NOTATIONS

λ_1 / λ_2	:	Rate of major/minor faults.
x / y	:	Probability of minor faults occur in non-redundant/redundant subsystem.
x_1 / y_1	:	Probability of major faults occur in non-redundant/redundant subsystem.
a / b	:	Probability of faults occur in a redundant subsystem of Type-I / Type-II.
$i_1(t)/I_1(t)$	:	P.d.f./c.d.f. of time to inspect the system.
$i_2(t)/I_2(t)$	:	P.d.f./c.d.f. of time to inspect the redundant subsystem.
$g(t)/G(t)$	:	P.d.f./c.d.f. of time to repair a major fault in the non-redundant subsystem.
$g_1(t)/G_1(t)$	:	P.d.f./c.d.f. of time to repair a minor fault in the non-redundant subsystem.
$g_2(t)/G_2(t)$	:	P.d.f./c.d.f. of time to repair a minor fault in Type-I redundant subsystem.
$g_3(t)/G_3(t)$	:	P.d.f./c.d.f. of time to repair a minor fault in Type-II redundant subsystem.

STATE TRANSITION DIAGRAM

A transition diagram in fig.1 shows the various states of transition. The epochs of entry in to state 0, 1, 3, 4, 5, 6, 7, 8 and 9 are regenerative point, i.e. these states are regenerative states while the state 2 is the non-regenerative states.

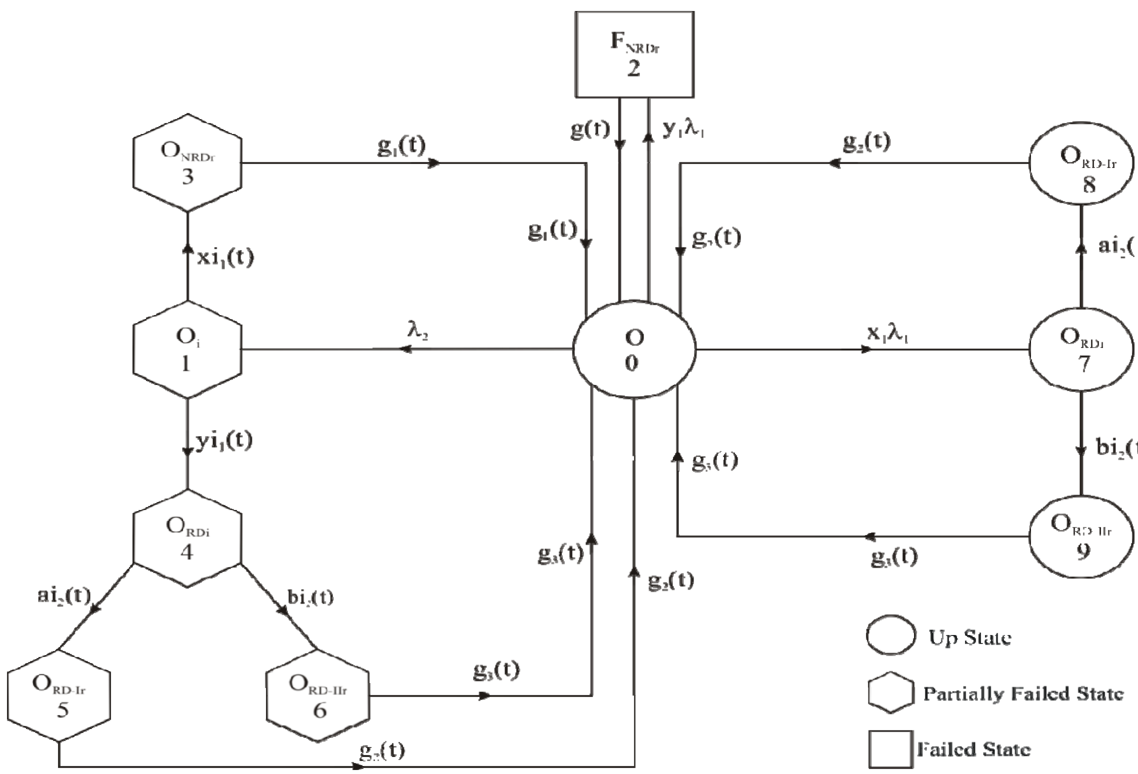

Fig.1

TRANSITION PROBABILITIES AND MEAN SOJOURN TIMES

The transition probabilities are obtained as under:

$$dQ_{01}(t) = \lambda_2 e^{-(\lambda_1+\lambda_2)t}dt \qquad dQ_{02}(t) = y_1\lambda_1 e^{-(\lambda_1+\lambda_2)t}dt$$

$$dQ_{07}(t) = x_1\lambda_1 e^{-(\lambda_1+\lambda_2)t}dt \qquad dQ_{13}(t) = xi_1(t)dt$$

$$dQ_{14}(t) = yi_1(t)dt \qquad dQ_{20}(t) = g(t)dt$$

$$dQ_{30}(t) = g_1(t)dt \qquad dQ_{45}(t) = ai_2(t)dt$$

$dQ_{46}(t) = bi_2(t)dt$ $\quad$ $dQ_{50}(t) = g_2(t)dt$

$dQ_{60}(t) = g_3(t)dt$ $\quad$ $dQ_{78}(t) = ai_2(t)dt$

$dQ_{79}(t) = bi_2(t)dt$ $\quad$ $dQ_{80}(t) = g_2(t)dt$

$dQ_{90}(t) = g_3(t)dt$

Taking Laplace Stieltjes Transformation $Q_{ij}^{**}(s)$ and $p_{ij} = \lim_{s \to 0} Q_{ij}^{**}(s)$, the non-zero elements $p_{ij,}$ computed are

$p_{01} = \dfrac{\lambda_2}{\lambda_1 + \lambda_2}$ $\quad$ $p_{02} = \dfrac{y_1\lambda_1}{\lambda_1 + \lambda_2}$ $\quad$ $p_{07} = \dfrac{x_1\lambda_1}{\lambda_1 + \lambda_2}$

$p_{13} = xi_1^{*}(0)$ $\quad$ $p_{13} = xi_1^{*}(0)$ $\quad$ $p_{20} = g^{*}(0)$

$p_{30} = g_1^{*}(0)$ $\quad$ $p_{45} = ai_2^{*}(0)$ $\quad$ $p_{46} = bi_2^{*}(0)$

$p_{50} = g_2^{*}(0)$ $\quad$ $p_{60} = g_3^{*}(0)$ $\quad$ $p_{78} = ai_2^{*}(0)$

$p_{79} = bi_2^{*}(0)$ $\quad$ $p_{80} = g_2^{*}(0)$ $\quad$ $p_{90} = g_3^{*}(0)$

By these transition probabilities, it can be verified that

$p_{01} + p_{02} + p_{07} = 1$, $\quad$ $p_{13} + p_{14} = 1$, $\quad$ $p_{45} + p_{46} = 1$,

$p_{78} + p_{79} = 1$, $\quad$ $p_{20} = p_{30} = p_{50} = p_{60} = p_{80} = p_{90} = 1$

The unconditional mean time taken by the system to transit for any regenerative state j, when it is counted from epoch of entrance into that state i, is mathematically stated as

$$m_{ij} = \int_0^\infty t dQ_{ij}(t) = -q_{ij}^{*\prime}(0),$$

Thus we get

$m_{01} + m_{02} + m_{07} = \mu_0$ $\quad$ $m_{13} + m_{14} = \mu_1$ $\quad$ $m_{20} = \mu_2$

$m_{30} = \mu_3$ $\quad$ $m_{45} + m_{46} = \mu_4$ $\quad$ $m_{50} = \mu_5$

$m_{60} = \mu_6$ $\quad$ $m_{78} + m_{79} = \mu_7$ $\quad$ $m_{80} = \mu_8$

$m_{90} = \mu_9$

The mean sojourn time in the regenerative state i(μi) is defined as the time of stay in that state before transition to any other state then we have

$\mu_0 = \dfrac{1}{\lambda_1 + \lambda_2}$ $\quad$ $\mu_1 = -i_1^{*\prime}(0)$ $\quad$ $\mu_2 = -g^{*\prime}(0)$

$\mu_3 = -g_1^{*\prime}(0)$ $\quad$ $\mu_4 = -i_2^{*\prime}(0)$ $\quad$ $\mu_5 = -g_2^{*\prime}(0)$

$\mu_6 = -g_3^{*\prime}(0)$ $\quad$ $\mu_7 = -i_2^{*\prime}(0)$ $\quad$ $\mu_8 = -g_2^{*\prime}(0)$

$\mu_9 = -g_3^{*\prime}(0)$

OTHER MEASURES OF SYSTEM EFFECTIVENESS

Using the arguments of the theory of regenerative processes, various measures of the system effectiveness obtained in steady state are as under:

Mean Time to System Failure (MTSF) **= N/D**

Expected Up-Time of the System with Full Capacity (AF_0) **= N_1/D_1**

Expected Up-Time of the System with Reduced Capacity (AR_0) **= N_2/D_1**

Busy Period of Repairman (Inspection time only) (B_i) **= N_3/ D_1**

Busy Period of Repairman (Repair time only) (B_r) = N_4 / D_1

where

$$N = \mu_0 + p_{01}\left[\mu_1 + p_{13}\mu_3 + p_{14}\left\{\mu_4 + p_{45}\mu_5 + p_{46}\mu_6\right\}\right] + p_{07}\left(\mu_7 + p_{78}\mu_8 + p_{79}\mu_9\right)$$

$$D = 1 - p_{01}\left[p_{13}p_{30} + p_{14}\left(p_{45}p_{50} + p_{46}p_{60}\right)\right] - p_{07}\left(p_{78}p_{80} + p_{79}p_{90}\right)$$

$$N_1 = \mu_0 + p_{07}\left(\mu_7 + p_{78}\mu_8 + p_{79}\mu_9\right)$$

$$D_1 = \mu_0 + p_{02}\mu_2 + p_{01}\left[\mu_1 + p_{13}\mu_3 + p_{14}\left\{\mu_4 + p_{45}\mu_5 + p_{46}\mu_6\right\}\right] + p_{07}\left(\mu_7 + p_{78}\mu_8 + p_{79}\mu_9\right)$$

$$N_2 = p_{01}\left[\mu_1 + p_{13}\mu_3 + p_{14}\left(\mu_4 + p_{45}\mu_5 + p_{46}\mu_6\right)\right]$$

$$N_3 = p_{01}\left(\mu_1 + p_{14}\mu_4\right) + p_{07}\mu_7$$

$$N_4 = p_{02}\mu_2 + p_{01}\left[p_{13}\mu_3 + p_{14}\left\{p_{45}\mu_5 + p_{46}\mu_6\right\}\right] + p_{07}\left(p_{78}\mu_8 + p_{79}\mu_9\right)$$

PROFIT ANALYSIS

The expected profit incurred of the system in the steady state is given by

$$P = C_0 AF_0 + C_1 AR_0 - C_2 B_i - C_3 B_r - C_4$$

where

C_0 = Revenue per unit uptime with full capacity of the system.

C_1 = Revenue per unit uptime with reduced capacity of the system.

C_2 = Cost per unit inspection of the subsystem

C_3 = Cost per unit repair of the subsystem

C_4 = Cost of installation of the system

GRAPHICAL INTERPRETATION

For graphical analysis purpose, following particular case is considered:

$g(t) = \beta e^{-\beta t}$ $\quad g_1(t) = \beta_1 e^{-\beta_1 t}$ $\quad g_2(t) = \beta_2 e^{-\beta_1 t}$

$g_3(t) = \beta_3 e^{-\beta_3 (t)}$ $\quad i_1(t) = \alpha_1 e^{-\alpha_1 t}$ $\quad i_2(t) = \alpha_2 e^{-\alpha_2 t}$

$p_{01} = \dfrac{\lambda_2}{\lambda_1 + \lambda_2}$ $\quad p_{02} = \dfrac{y_1\lambda_1}{\lambda_1 + \lambda_2}$ $\quad p_{07} = \dfrac{x_1\lambda_1}{\lambda_1 + \lambda_2}$

$p_{13} = x$ $\quad p_{14} = y$ $\quad p_{20} = 1$

$p_{46} = b = p_{79}$ $\quad \mu_0 = \dfrac{1}{\lambda_1 + \lambda_2}$ $\quad \mu_1 = \dfrac{1}{\alpha_1}$

$\mu_2 = \dfrac{1}{\beta}$ $\quad \mu_3 = \dfrac{1}{\beta_1}$ $\quad \mu_4 = \dfrac{1}{\alpha_2}$

$\mu_5 = \dfrac{1}{\beta_2}$ $\quad \mu_6 = \dfrac{1}{\beta_3}$ $\quad \mu_7 = \dfrac{1}{\alpha_2}$

$\mu_8 = \dfrac{1}{\beta_2}$ $\quad \mu_9 = \dfrac{1}{\beta_3}$

Various graphs are plotted for MTSF, expected uptimes with full/reduced capacity and profit of the system taking several values of rates of different faults (λ_1, λ_2), repair rates (β, β_1, β_2, β_3), inspection rates (α_1, α_2) and various probabilities (a, b, x, y, x_1 & y_1).

Following is concluded from the various plotted graphs:

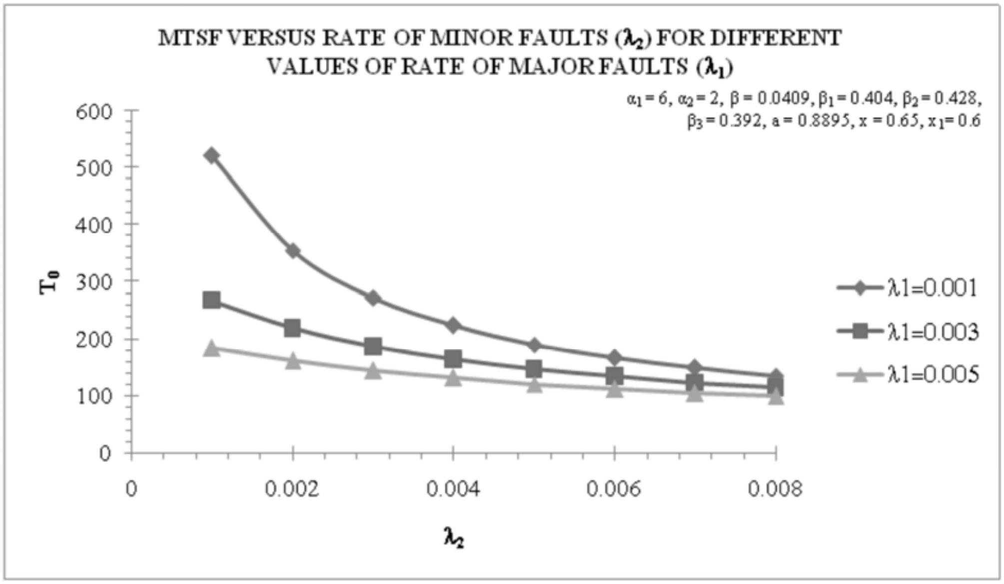

Fig.2

Fig.2 gives the graph between MTSF (T_0) and rate of minor faults (λ_2) for different values of rate of major faults (λ_1). It is concluded that that the MTSF decreases with increase in values of the rate of minor fault. Also MTSF has lower values with the higher values of the rate of major fault.

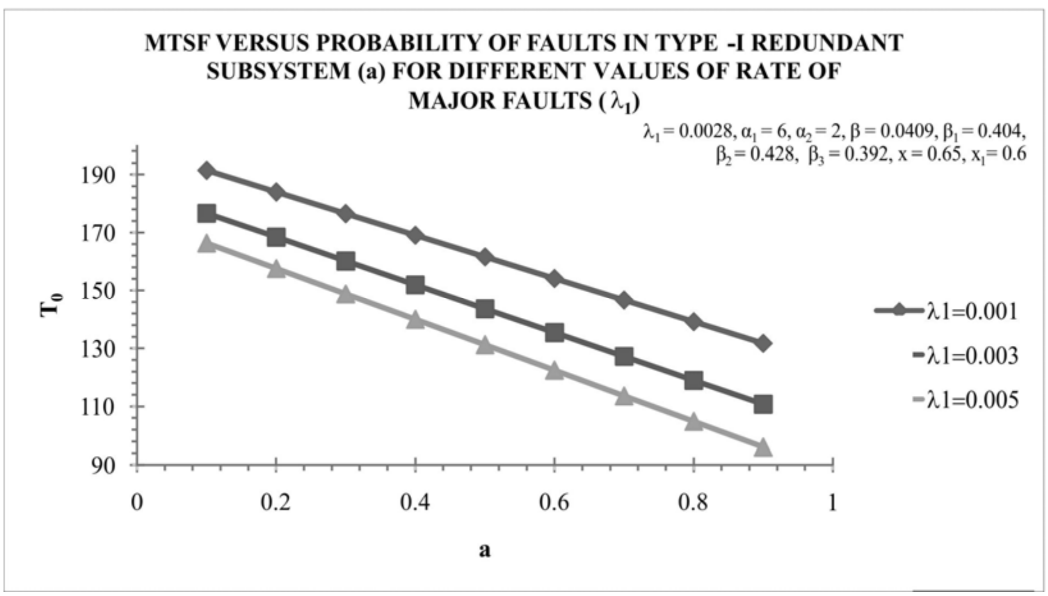

Fig.3

Fig.3 gives the graph between MTSF (T_0) and probability of faults in type-I redundant subsystem (a) for different values of rate of major faults (λ_1). The graph reveals that the MTSF decreases with increase in values of the probability of a fault in type-I redundant subsystem. Also MTSF has lower values with the higher values of the rate of major faults.

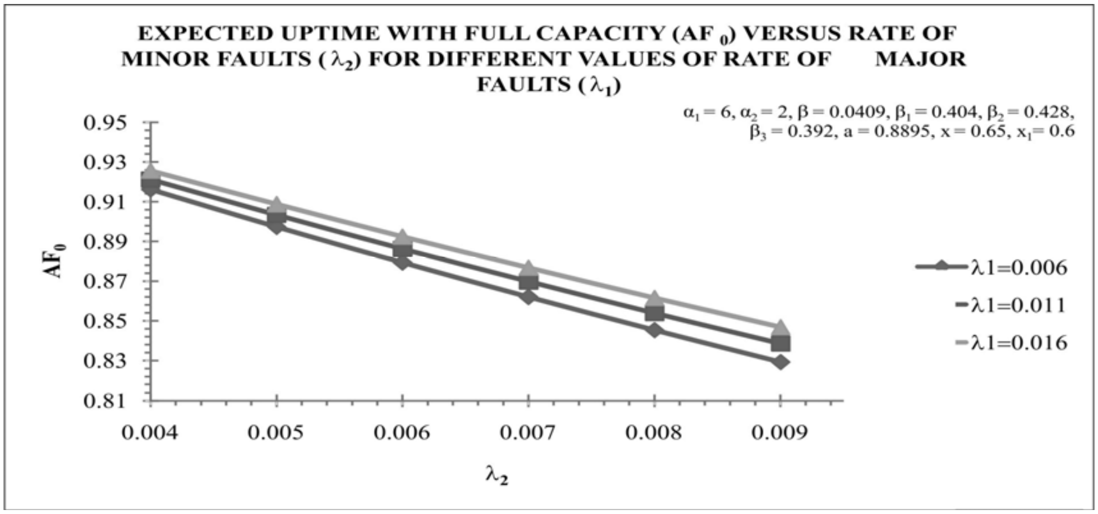

Fig.4

Fig.4 presents the graph between expected uptime of the system with full capacity (AF_0) and rate of minor faults (λ_2) for different values of rate of major faults (λ_1). It can be concluded that the expected uptime of the system with full capacity decreases with increase in values of the rate of minor faults while it has higher values with the higher values of the rate of major faults.

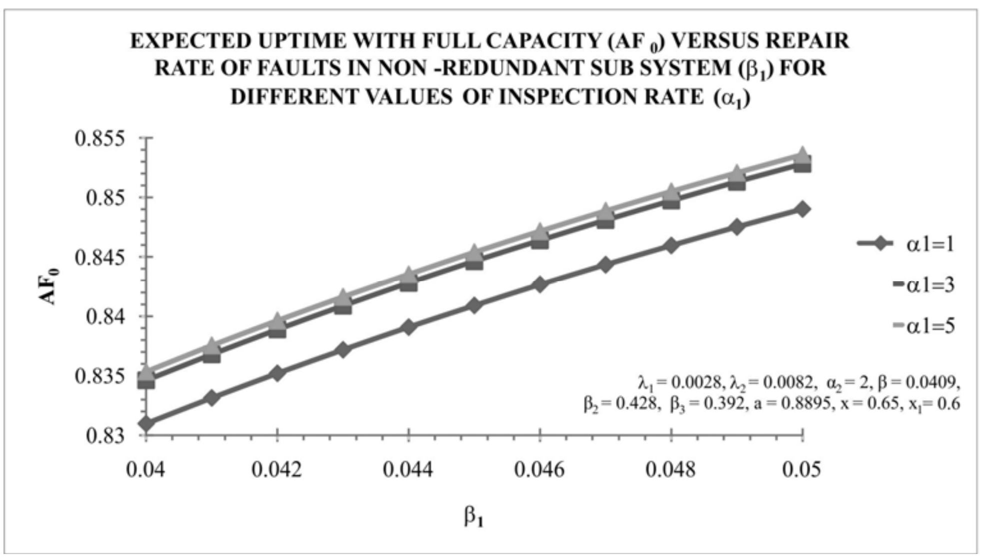

Fig.5

Fig.5 gives the graph of expected uptime of the system with full capacity (AF_0) with repair rate of the faults in non-redundant subsystem (β_1) for different values of inspection rate (α_1). The graph reveals that the expected uptime of the system with full capacity increases with increase in values of the repair rate of the faults in non-redundant subsystem and it has higher values with the higher values of the inspection rate.

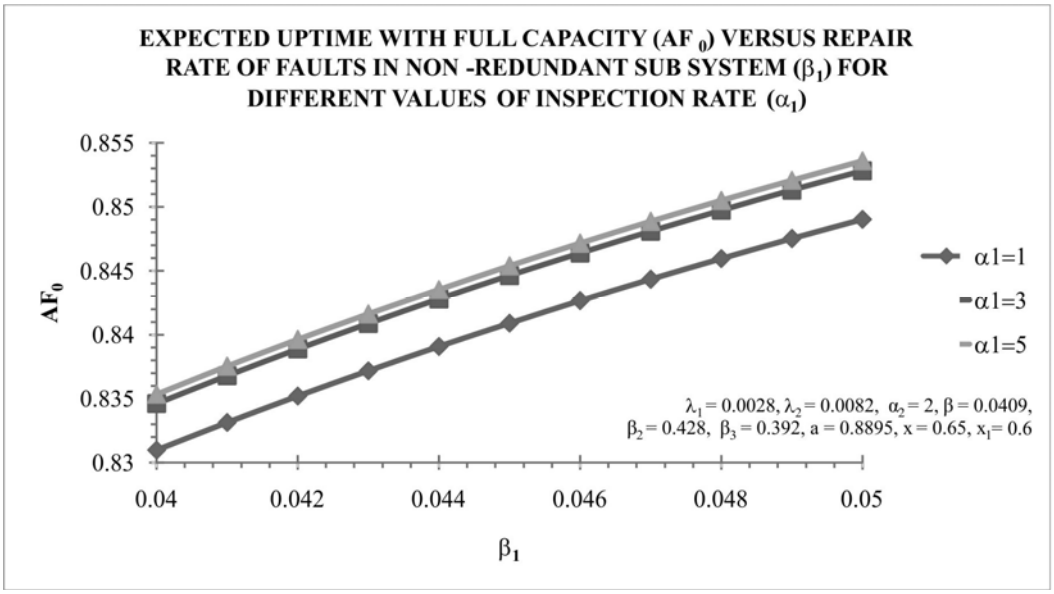

Fig.6

The graph in fig.6 gives patterns of expected uptime of the system with full capacity with repair rate of the faults in type-II redundant subsystem (β_3) for different values of the probability of major faults in non-redundant subsystem (x_1). From the graph it can be concluded that the expected uptime of the system with full capacity increases with increase in values of the repair rate of the faults in type-II redundant subsystem and it has higher values with the higher values of the probability of major fault in non-redundant subsystem.

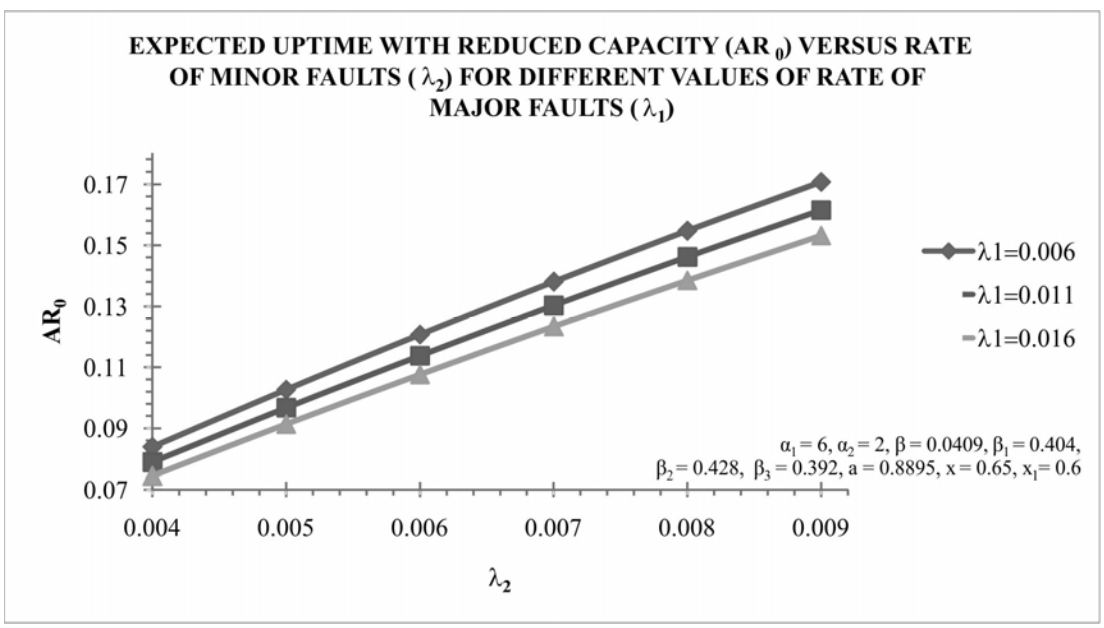

Fig.7

Fig.7 gives the graph between expected uptime of the system with reduced capacity (AR_0) and rate of minor faults (λ_2) for different values of the failure rate (λ_1) due to major faults. The graph reveals that the expected uptime of the system with reduced capacity increases with increase in values of the rate of minor faults while it has lower values with the higher values of the rate of major faults.

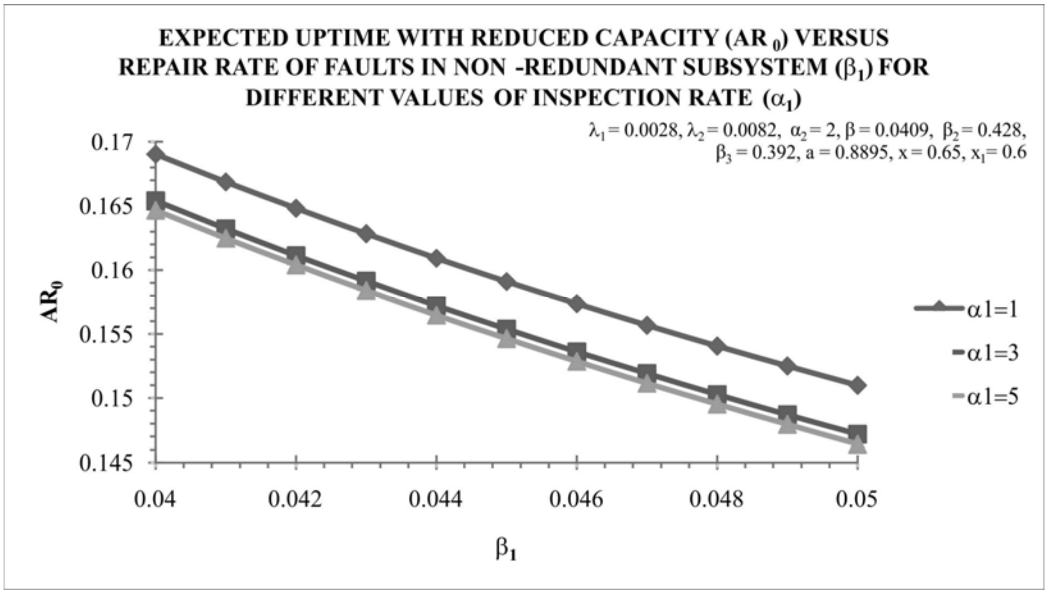

Fig.8

The curves in fig.8 reveal the behaviour of expected uptime of the system with reduced capacity (AR_0) with respect to repair rate of faults in non-redundant subsystem (β_1) for different values of inspection rate (α_1). It can be concluded from the graph that the expected uptime of the system with reduced capacity decreases with increase in values of the repair rate of faults in non-redundant subsystem and also this expected uptime of the system has lower values with the higher values of the inspection rate.

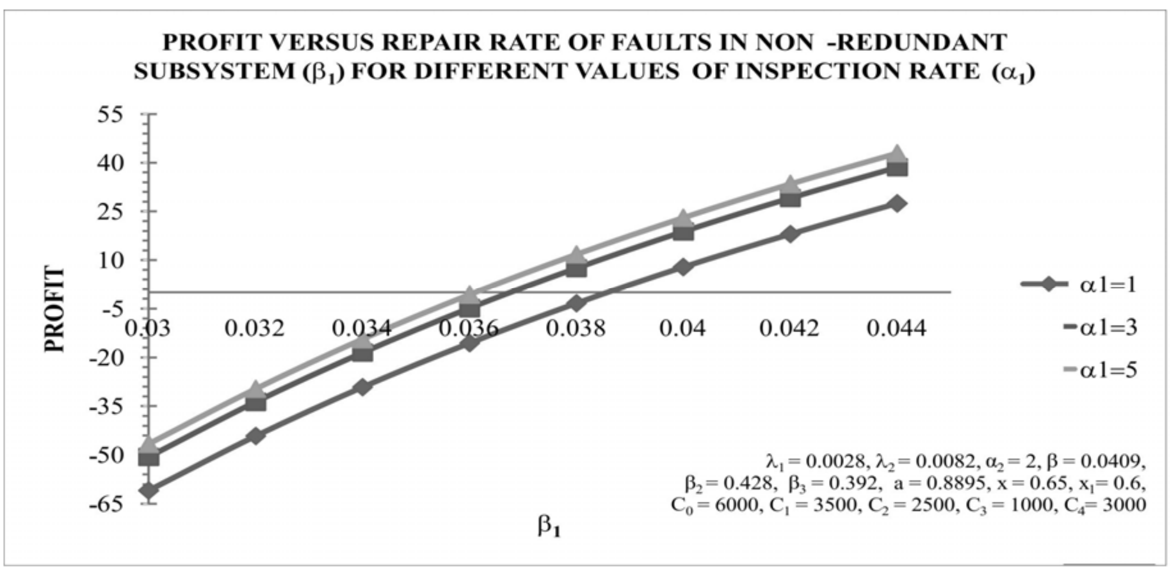

Fig.9

The curves in the fig.9 show the behaviour of profit of the system (P) with respect to repair rate of faults in non-redundant subsystem (β_1) for different values of inspection rate (α_1). It is evident from the graph that the profit increases with the increase in the values of repair rate of faults in non-redundant subsystem and has higher values for higher values of inspection rate when other parameters remain fixed. From the fig.9, it can also be observed that for $\alpha_1 = 1$, the profit is negative or zero or positive according as $\beta_1$is < or = or > 0.0386. Thus, in this case, the system is profitable whenever $\beta_1 > 0.0386$. Similarly, for $\alpha_1 = 3$ and 5 the system is profitable whenever $\beta_1 > 0.0367$ and 0.0361 respectively.

The curves in the fig.10 show the behaviour of profit (P) with respect to revenue per unit uptime of the system with full capacity (C_0) for different values of the rate of minor faults (λ_2). It is concluded from the graph that the profit increases with the increase in the values of revenue per unit uptime of the system with full capacity and has lower values for higher values of the rate of minor faults when other parameters remain fixed. From the fig.10 it can also be observed that for $\lambda_2 = 0.001$, the profit is negative or zero or positive according as C_0 is < or = or > 3558.13. Thus in this case, the system is profitable whenever $C_0 > 3558.13$. Similarly, for $\lambda_2 = 0.004$ and 0.007, the system is profitable respectively whenever $C_0 > 3667.83$ and 3777.53.

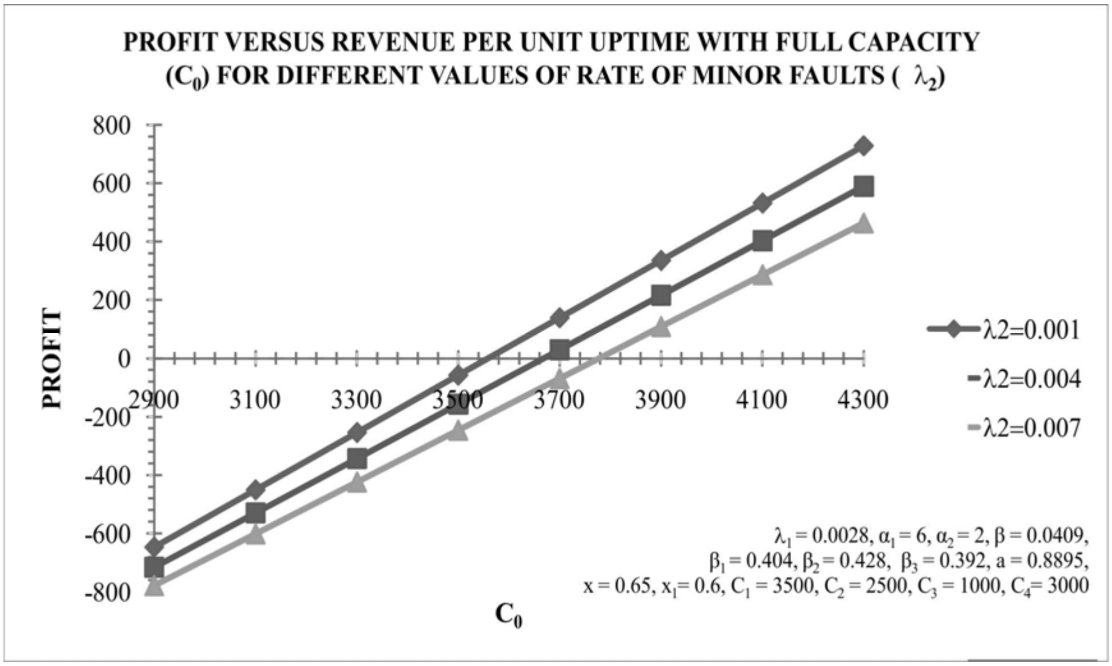

Fig.10

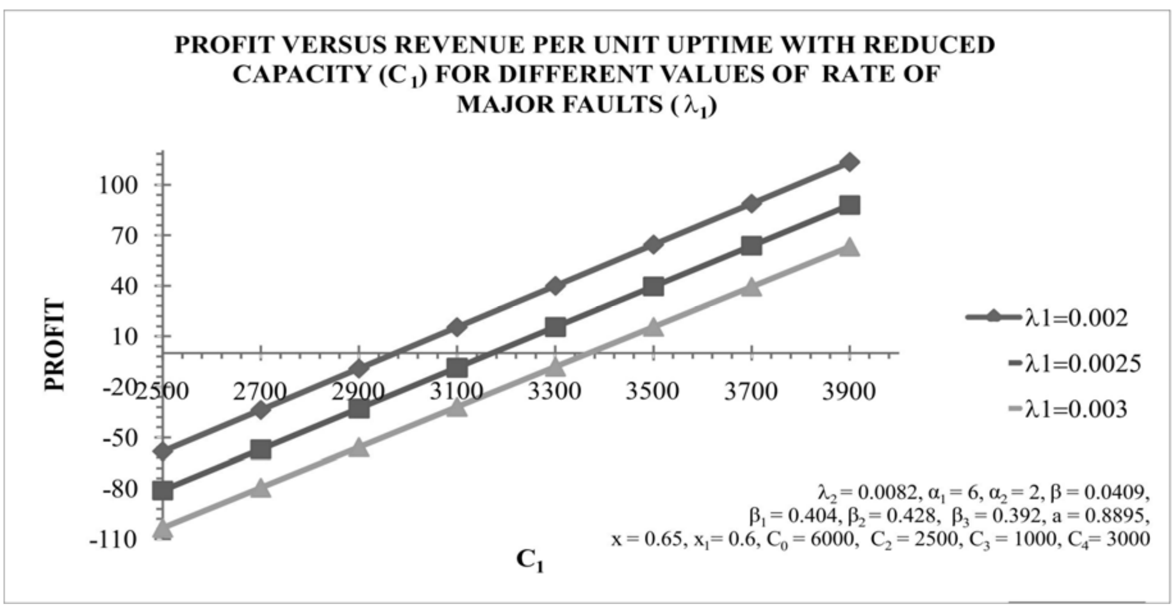

Fig.11

The curves in the fig.11 show the behaviour of profit (P) with respect to revenue per unit uptime of the system with reduced capacity (C_1) for different values of the rate of major faults (λ_1). It is concluded from the graph that the profit increases with the increase in the values of revenue per unit uptime of the system with reduced capacity and has lower values for higher values of the rate of minor faults when other parameters remain fixed. From the fig.11 it can also be observed that for $\lambda_1 = 0.001$, the profit is negative or zero or positive according as C_1 is < or = or > 2975.75. Thus in this case, the system is profitable whenever $C_1 > 2975.75$. Similarly, for $\lambda_1 = 0.004$ and 0.007, the system is profitable whenever $C_1 > 7172.47$ and 3369.19 respectively.

CONCLUSION

A stochastic model for industrial process water system is developed incorporating the realistic and practical aspects of automatic switching of components in redundant subsystem on occurrence of major faults however switching of the redundant subsystems on occurrence of a minor fault is not carried out to avoid losses. From the graphical analysis, it can be concluded that the reliability, expected uptime and profit of the water process system have lower values with the higher values of the rates of minor / major faults. Moreover the reliability, expected uptime and profit of the system have higher values with the higher values of the various repair and inspection rates.

Various cut-off points for the profit of the system are obtained with respect to the rates of minor/major faults, repair, inspection and revenue per unit up time of the system working with full capacity and reduced capacity. The analysis on the basis of real data will help all the stakeholders of the system in the industry and society to evaluate the performance of the system and to decide its various operational costs and revenues in the different working conditions of the system.

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